6.1-6.2: Orthogonality and Projection

Tuesday, October 11

Warmup

Define $\mathbf{u} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -1 & 2 & -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$. Find the following:

1. $\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$.
2. $\|\mathbf{v}\| = \sqrt{6}$
3. $\|\mathbf{w}\| = \sqrt{3}$
4. $\|\mathbf{u} + 2\mathbf{v}\|^2 = \|(-1, 6, 1)\|^2 = \sqrt{38}$
5. $\|\mathbf{u}\|^2 + 4\|\mathbf{v}\|^2 = 14 + 4 \cdot 6 = 38$
6. $\| - 3\mathbf{u}\| = 3\|\mathbf{u}\| = 3\sqrt{6}$
7. $\|\mathbf{u} + \mathbf{w}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{w}\|^2 = 14 + 3 = 17$
8. $\|\mathbf{v} + \mathbf{w}\|^2 = \|(0, 3, -2)\|^2 = \sqrt{13}$, note that $\mathbf{v}$ and $\mathbf{w}$ are NOT perpendicular!
9. $\mathbf{v} \cdot (\mathbf{u} - 2\mathbf{w}) = (-1)(-1) + 2(0) + (-1)(5) = -4$
10. $\mathbf{v} \cdot \mathbf{u} - 2\mathbf{v} \cdot \mathbf{w} = 0 - 2\mathbf{v} \cdot \mathbf{w} = -4$

Describe the shortest path from a point to a line.
ANSWER: the shortest path is perpendicular to the line.

Orthogonal Complements

If $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 0 & 1 \\ 0 & -3 \end{bmatrix}$, find a basis for $(Col(A))^{\perp}$.

ANSWER: $A^T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 1 & -3 \end{bmatrix}$, which is already in echelon form. We can then set x_3 and x_4 as free variables, and get

$$\mathbf{x} = \begin{bmatrix} x_2 \\ (3x_4 - x_3)/2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 + \frac{3}{2}x_4 \\ -\frac{1}{2}x_3 + \frac{3}{2}x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \frac{3}{2} & \\ \frac{3}{2} & \\ 0 & 1 \end{bmatrix}.$$

The orthogonal complement is then the span of these two vectors. Not that this uses the fact that $(Col(A))^{\perp} = Nul(A^T)$.

Projections

Define $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, and let $\hat{\mathbf{y}}$ be the projection of $\mathbf{y}$ onto the span of $\mathbf{b}_1$.

1. Verify that $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal.

   ANSWER: $\mathbf{b}_1 \cdot \mathbf{b}_2 = 1(1) + 1(-1) = 0$.

2. If $\mathbf{y} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2$, find an expression for $\hat{\mathbf{y}}$ in terms of the c_i and $\mathbf{b}_i$.

   ANSWER: $\hat{\mathbf{y}} = c_1\mathbf{b}_1$, since $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal.

3. What are c_1 and c_2 in terms of $\mathbf{b}_1 \cdot \mathbf{y}$ and $\mathbf{b}_2 \cdot \mathbf{y}$?

   ANSWER: $c_1 = \mathbf{b}_1 \cdot \mathbf{y}/\|\mathbf{b}_1\|^2$ and $c_2 = \mathbf{b}_2 \cdot \mathbf{y}/\|\mathbf{b}_2\|^2$.

4. Sketch a triangle whose vertices are the origin, $\mathbf{y}$, and $\hat{\mathbf{y}}$. Label the lengths of the sides, and find an expression for $\cos(\mathbf{y}, \mathbf{b}_1)$. Label the appropriate angle.

   PICTURE.

5. Find c_1 and c_2.

   ANSWER: $c_1 = 4.5, c_2 = -1.5$.

6. Show that $\mathbf{y} - \hat{\mathbf{y}}$ and $\hat{\mathbf{y}}$ are orthogonal.

   ANSWER: $\hat{\mathbf{y}} = c_1\mathbf{b}_1$ and $\mathbf{y} - \hat{\mathbf{y}} = c_2\mathbf{b}_2$. These are orthogonal since $\mathbf{b}_1$ and $\mathbf{b}_2$ are orthogonal.

Orthogonal Matrices

Find a matrix P such that $P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_1 \\ x_2 \end{bmatrix}$. Find expressions for $\|\mathbf{x}\|^2$ and $\|P\mathbf{x}\|^2$. Is P orthogonal?

ANSWER: $P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. $\|\mathbf{x}\|^2 = \|P\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2$. P is orthogonal because $P^TP = I$. One consequence is that $\|\mathbf{x}\| = \|P\mathbf{x}\|$ for all $\mathbf{x}$, but this condition does not necessarily mean that P is orthogonal on its own.