6.3-6.4: Projection and Gram-Schmidt

Thursday, October 13

Projection

Find the point on the line that passes through (0,0) and (3,1) closest to the point (5,5). Draw a picture.
ANSWER: A vector spanning the given line is (3,1), so we can use the formula $\hat{\mathbf{b}} = \frac{\mathbf{a}\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|^2}$ and get that the projection is equal to $\begin{bmatrix}3\\1\end{bmatrix}\frac{20}{10} = \begin{bmatrix}6\\2\end{bmatrix}$.

If Π is the matrix that projects a point onto the line mentioned above, find Π explicitly. What is its eigendecomposition?
ANSWER: $\Pi = \frac{1}{10}\begin{bmatrix}3\\1\end{bmatrix}\begin{bmatrix}3 & 1\end{bmatrix} = \begin{bmatrix}.9 & .3\\.3 & .1\end{bmatrix}$. Its characteristic polynomial is $\lambda^2 - \lambda$ and so it has eigenvalues 0 and 1. The eigenvectors are $\begin{bmatrix}3\\1\end{bmatrix}$ with eigenvalue 1 and $\begin{bmatrix}-1\\3\end{bmatrix}$ with eigenvalue 0.

Gram-Schmidt

Find an orthogonal basis for Span $\left\{\begin{bmatrix}0\\4\\2\end{bmatrix}, \begin{bmatrix}5\\6\\-7\end{bmatrix}\right\}$.

ANSWER: Divide the first vector (call it $\mathbf{v}_1$) by its norm to get $\begin{bmatrix}0\\\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{bmatrix}$. The inner product of this vector with $\mathbf{v}_2$ is $\sqrt{5}$, so subtracting $\sqrt{5}$ from $\mathbf{v}_2$ gives an orthogonal vector $\begin{bmatrix}5\\4\\-8\end{bmatrix}$. This vector has norm $\sqrt{105}$, so divide by the norm to get a second orthogonal vector of $\frac{1}{\sqrt{105}}\begin{bmatrix}5\\4\\-8\end{bmatrix}$.

True/False

1. The sum of two orthogonal matrices is an orthogonal matrix. FALSE: the identity matrix I is orthogonal but $I + I = 2I$ is not.
2. The product of two orthogonal matrices is an orthogonal matrix. TRUE: If U and V are orthogonal then $(UV)^T(UV) = V^TU^TUV = V^TIV = V^TV = I$.
3. If U is orthogonal then $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for any $\mathbf{x}$. TRUE: $\|U\mathbf{x}\|^2 = \mathbf{x}^TU^TU\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$.
4. If U is orthogonal the the angle between $U\mathbf{x}$ and $U\mathbf{y}$ is the same as the angle between $\mathbf{x}$ and $\mathbf{y}$ for any $\mathbf{x}$ and $\mathbf{y}$. TRUE: In general we know that if θ is the angle between $\mathbf{x}$ and $\mathbf{y}$ then $\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\|\mathbf{y}\|\cos\theta$. So
$$\cos\theta(U\mathbf{x}, U\mathbf{y}) = \frac{(U\mathbf{x})^T(U\mathbf{y})}{\|U\mathbf{x}\|\|U\mathbf{y}\|} = \frac{\mathbf{x}^T\mathbf{y}}{\|\mathbf{x}\|\|\mathbf{y}\|} = \cos\theta(\mathbf{x}, \mathbf{y})$$

5. If A and B are both diagonalizable and have the same eigenvectors then $AB = BA$. TRUE: If $A = PD_AP^{-1}$ and $B = PD_BP^{-1}$ then

$$AB = PD_AP^{-1}PD_BP^{-1} = PD_AD_BP^{-1} = PD_BD_AP^{-1} = PD_BP^{-1}PD_AP^{-1} = BA.$$

6. If $A^2 = A$ then all eigenvalues of A are 0 or 1. TRUE: If $A\mathbf{x} = \lambda\mathbf{x}$ then $\lambda^2\mathbf{x} = A^2\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}$, so $\lambda^2 = \lambda$ and therefore $\lambda = 0$ or $\lambda = 1$.

7. If all eigenvalues of A are 0 or 1 then $A^2 = A$. FALSE: try $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

8. For all $\mathbf{y}$ and each subspace W the vector $\mathbf{y} - \text{proj}_W \mathbf{y}$ is orthogonal to W. TRUE: If U is an orthogonal basis for W then

$$U^T(\mathbf{y} - \hat{\mathbf{y}}) = U^T(\mathbf{y} - UU^T\mathbf{y}) = U^T\mathbf{y} - U^TUU^T\mathbf{y} = U^T\mathbf{y} - U^T\mathbf{y} = 0.$$

9. For any $\mathbf{y} \in \mathbb{R}^n$ and any subspace $W \subset \mathbb{R}^n$, $\text{proj}_W(\text{proj}_W \mathbf{y}) = \text{proj}_W \mathbf{y}$. TRUE: $UU^TUU^T\mathbf{y} = U(U^TU)U^T\mathbf{y} = UU^T\mathbf{y}$.