5.2-5.3: Eigenvalues and Diagonalization

Tuesday, October 4

Warmup

True/False:

1. Eigenvectors with distinct eigenvalues are linearly independent. TRUE.
2. The set of eigenvectors of a matrix A with a particular eigenvalue λ forms a subspace. TRUE.
3. If A and B are row equivalent then A and B have the same determinant. FALSE. Adding a multiple of one row to another does not change the determinant but scaling and swapping rows do.
4. If A and B are row equivalent then A and B have the same null space. TRUE.

Find the determinant of $\begin{bmatrix} 3-\lambda & 2 \\ 0 & -\lambda \end{bmatrix}$ in terms of λ. When is it equal to zero?
ANSWER: the matrix is triangular so the determinant is equal to $(3-\lambda)(-\lambda)$, which is zero when $\lambda = 0$ or $\lambda = 3$.

If $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ and Λ is a diagonal matrix with $\{\lambda_1, \ldots, \lambda_n\}$ as its diagonal elements, what is $V\Lambda$?
ANSWER: $V\Lambda = [\lambda_1\mathbf{v}_1, \ldots, \lambda_n\mathbf{v}_n]$.

Diagonalizing a Matrix

Let A be the matrix $\begin{bmatrix} -4 & -8 \\ 3 & 7 \end{bmatrix}$. Find the characteristic polynomial of A.
ANSWER: $\det \begin{bmatrix} -4-\lambda & -8 \\ 3 & 7-\lambda \end{bmatrix} = (-4-\lambda)(7-\lambda) + 24 = \lambda^2 - 3\lambda - 4$.

Find the eigenvalues of A, and find eigenvectors for those eigenvalues.
ANSWER: the eigenvalues of A are the zeroes of the characteristic polynomial, which are 4 and -1 since $\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$.
The eigenvectors of A are then non-trivial elements of the null spaces of $A - 4I$ and $A + I$, or $\begin{bmatrix} -8 & -8 \\ 3 & 3 \end{bmatrix}$ and $\begin{bmatrix} -3 & -8 \\ 3 & 8 \end{bmatrix}$. Solving this gives that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ has eigenvalue 4 and $\begin{bmatrix} -8 \\ 3 \end{bmatrix}$ has eigenvalue -1. (Verify this!)

Verify that the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, and therefore that $V = [\mathbf{v}_1, \mathbf{v}_2]$ is invertible.

$$V^{-1} = \begin{bmatrix} -1 & -8 \\ 1 & 3 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 8 \\ -1 & -1 \end{bmatrix}.$$

If $AV = V\Lambda$ and V is invertible, then $V^{-1}AV = \Lambda$. Verify that this is true.

$$
\begin{aligned}
V^{-1}AV &= \frac{1}{5}\begin{bmatrix} 3 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -4 & -8 \\ 3 & 7 \end{bmatrix}\begin{bmatrix} -1 & -8 \\ 1 & 3 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 3 & 8 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -4 & 8 \\ 4 & -3 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}
\end{aligned}
$$

Find A^4 in terms of Λ and V. Show that your formula is correct.

$$
\begin{aligned}
A^4 &= (V\Lambda V^{-1})^4 \\
&= V\Lambda V^{-1}V\Lambda V^{-1}V\Lambda V^{-1}V\Lambda V^{-1} \\
&= V\Lambda I\Lambda I\Lambda I\Lambda V^{-1} \\
&= V\Lambda^4 V^{-1}.
\end{aligned}
$$

Use your previous answer to find A^4.

$$
\begin{aligned}
A^4 &= V\Lambda^4 V^{-1} \\
&= \frac{1}{5}\begin{bmatrix} -1 & -8 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 8 \\ -1 & -1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} -1 & -8 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 768 & 2048 \\ -1 & -1 \end{bmatrix} \\
&= \frac{1}{5}\begin{bmatrix} -760 & -2040 \\ 765 & 2045 \end{bmatrix} \\
&= \begin{bmatrix} -152 & -408 \\ 153 & 409 \end{bmatrix}
\end{aligned}
$$

Was that actually easier then just finding A^4 directly? Probably not, but for much higher powers and larger matrices it would be. More important is the bigger picture of what's going on: the part with eigenvalue 4 is overwhelming the part with eigenvalue -1.

Complex Numbers and Eigenvalues

Let i be a number such that $i^2 = -1$. Simplify:

1. $i^{10} + 3i^3 = -1 - 3i$
2. $(3+i)(1-2i) = 5 - 5i$
3. $\overline{2+i} = 2 - i$
4. $\frac{1}{1-i} = \frac{1+i}{(1+i)(1-i)} = \frac{1+i}{2}$.

If $A^2 = 0$, show that 0 is the only possible eigenvalue of A. Does A have to be the zero matrix?
If there is some $\lambda \neq 0$ with eigenvector $\mathbf{v}$, then $\lambda^2\mathbf{v} = \lambda(A\mathbf{v}) = A^2\mathbf{v} = 0 \cdot \mathbf{v} = \mathbf{0}$, so $\lambda^2 = 0$ and therefore $\lambda = 0$.
A does not have to be the zero matrix: for example, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

If $A^2 = -I$, is it possible for A to have real eigenvalues?
ANSWER: No. If $\mathbf{v}$ has eigenvalue λ, then applying both sides to $\mathbf{v}$ imply that $\lambda^2\mathbf{v} = A^2\mathbf{v} = -\mathbf{v}$, so $\lambda^2 = -1$.