5.3-5.4: More Eigenvectors, More Diagonalization

Tuesday, October 6

Warmup

Decide whether each statement *and its converse* are True or False. Assume A is an $n \times n$ matrix.

1. If A has n linearly independent eigenvectors then it is diagonalizable.
2. If A is diagonalizable then it has n distinct eigenvalues.
3. If A is invertible then it is diagonalizable.
4. If A is invertible then all of its eigenvalues are nonzero.

Define $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find $\lim_{n\to\infty} A^n x_i$ for $i = 1, 2$ for each of the following matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

Miscellany

If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, find the eigenvalues of A $2A$, and $A^2 - A/2$. Come up with a conjecture.

If $A^2 = 0$, what can you say about the eigenvalues of A? What if $A^2 = A$? If $A \neq I$ but $A^2 = I$?

If $T : \mathbb{P}_3 \to \mathbb{P}_2$ is given by $T(p(t)) = p(t) - p'(t)$, find a matrix representation for T given the bases $\{1, t, t^2, t^3\}$ and $\{1, t, t^2\}$.

Obligatory Application: The Fibonacci Sequence

Define: $f_{-1} = 1, f_0 = 0$, and for all $n \geq 1, f_n = f_{n-1} + f_{n-2}$. Find f_1 through f_7.

There is a matrix A such that $\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = A \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix}$. Find A.

Find the eigenvalues and eigenvectors of A. Call this basis $\mathcal{B}$.

If $\mathbf{x}_0 = \begin{bmatrix} f_0 \\ f_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find $[\mathbf{x}_0]_{\mathcal{B}}$ and $[A]_{\mathcal{B}}$.

Use this information to find a closed (non-recursive) formula for f_n, the n-th Fibonacci number.