5.3-5.4: More Eigenvectors, More Diagonalization

Tuesday, October 6

Warmup

Decide whether each statement *and its converse* are True or False. Assume A is an $n \times n$ matrix.

1. If A has n linearly independent eigenvectors then it is diagonalizable. TRUE. Converse is TRUE.
2. If A is diagonalizable then it has n distinct eigenvalues. FALSE. Converse is TRUE.
3. If A is invertible then it is diagonalizable. FALSE and FALSE.
4. If A is invertible then all of its eigenvalues are nonzero. TRUE and TRUE.

Define $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find $\lim_{n\to\infty} A^n x_i$ for $i = 1, 2$ for each of the following matrices:

$$\begin{bmatrix} 2 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

ANSWER: Case 1: limits are ∞ and 0. Case 2: $A^n\mathbf{x}_1 = \mathbf{x}_1$ for all n. $A^n\mathbf{x}_2$ will oscillate between $\mathbf{x}_2$ and $-\mathbf{x}_2$. Case 3: Both will grow to ∞ as n grows, since both have components in the eigenspace with $\lambda > 1$.

Miscellany

If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, find the eigenvalues of A $2A$, and $A^2 - A/2$. Come up with a conjecture.
ANSWER: in general, for a function f it should hold that $\lambda(f(A)) = f(\lambda)$. (Some nuance, depending on how you want to define $f(A)$.)

If $A^2 = 0$, what can you say about the eigenvalues of A? What if $A^2 = A$? If $A \neq I$ but $A^2 = I$?
ANSWER: Based on the previous part, we can say that $\lambda^2 = 0$ and so $\lambda = 0$ if $A^2 = 0$.
Case 2: If $A^2 = A$ then $\lambda^2 = \lambda$, so all eigenvalues of A are 0 or 1.
Case 3: if $A \neq I$ but $A^2 = I$ then all eigenvalues are ± 1, and since $A \neq I$ *at least one* of them is -1.

If $T : \mathbb{P}_3 \to \mathbb{P}_2$ is given by $T(p(t)) = p(t) - p'(t)$, find a matrix representation for T given the bases $\{1, t, t^2, t^3\}$ and $\{1, t, t^2\}$.
ANSWER:

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Obligatory Application: The Fibonacci Sequence

Define: $f_{-1} = 1, f_1 = 0$, and for all $n \geq 1, f_n = f_{n-1} + f_{n-2}$. Find f_1 through f_7.
ANSWER: $f_n = 0, 1, 1, 2, 3, 5, 8, 13, \ldots$

There is a matrix A such that $\begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = A \begin{bmatrix} f_{n-1} \\ f_{n-2} \end{bmatrix}$. Find A.
ANSWER: $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Find the eigenvalues and eigenvectors of A. Call this basis $\mathcal{B}$.
ANSWER: A has eigenvalues $\frac{1 \pm \sqrt{5}}{2}$ and eigenbasis $V = \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & -\frac{1+\sqrt{5}}{2} \end{bmatrix}$ (the columns are the eigenvectors).

If $\mathbf{x}_0 = \begin{bmatrix} f_0 \\ f_{-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, find $[\mathbf{x}_0]_{\mathcal{B}}$ and $[A]_{\mathcal{B}}$.
ANSWER: Because $\mathcal{B}$ is a basis of eigenvalues, $[A]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$.
Then $[\mathbf{x}_0]_{\mathcal{B}} = V^{-1}\mathbf{x}_0 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Use this information to find a closed (non-recursive) formula for f_n, the n-th Fibonacci number.
ANSWER:

$$\begin{aligned} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} &= A^n \mathbf{x}_0 & &= V(V^{-1}A^nVV^{-1}\mathbf{x}_0) \\ &= V\left(\frac{1}{\sqrt{5}} \begin{bmatrix} \lambda_1^n \\ -\lambda_2^n \end{bmatrix}\right) \\ &= \begin{bmatrix} \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} \\ \bigstar \end{bmatrix}. \end{aligned}$$

This formula gives that $f_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}}$, where $\lambda_1, \lambda_2 = \frac{1 \pm \sqrt{5}}{2}$. You can check that this formula obeys the same recurrence relation as the f_n.