4.4-4.6: Bases, Dimension, and Rank

Tuesday, September 27

Warmup

Given that B is an invertible matrix, which one of these is not like the others: Col(A), Col(BA), Col(AB)?

If you have two sets A and B such that $A \subseteq B$ and $B \subseteq A$, what can you conclude about the sets?

If you have a subspace $H \subset V$ such that $\dim(H) = \dim(V) = n$, what can you do to show that $H = V$?

Back to Bases

Let $\mathbb{P}_2$ be the set of polynomials of degree at most 2. Define $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{\frac{t^2-t}{2}, 1 - t^2, \frac{t^2+t}{2}\}$.

1. Express t^2 in terms of both $\mathcal{B}$ and $\mathcal{C}$.

2. If p is a polynomial such that $p(-1) = 1, p(0) = 0$ and $p(1) = 1$, express p in terms of both $\mathcal{B}$ and $\mathcal{C}$ (Use the fact that if $p = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$ then $p(x) = c_1\mathbf{b}_1(x) + c_2\mathbf{b}_2(x) + c_3\mathbf{b}_3(x)$ for all x, then solve the resulting linear system.)

3. Let $\mathbb{P}_2$ be the set of polynomials of degree at most 2. Prove that if $T(p) = \{p(-1), p(0), p(1)\}$ then $T : \mathbb{P}_2 \to \mathbb{R}^3$ is a linear transformation. Is it an isomorphism? Justify your answer.

4. Let $S \subseteq \mathbb{P}_2$ be the set of polynomials $p \in \mathbb{P}_2$ such that $p(3) = 1$. Find a basis for S and show that it is isomorphic to $\mathbb{R}^2$.

Rank

Let $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and take as given that they are row equivalent. What is rank(A)? What is dim Nul(A)? Find basis for the column, row, and null spaces of A.

If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors, prove that $A = \mathbf{u}\mathbf{v}^T$ is a rank-1 matrix.