4.4-4.6: Bases, Dimension, and Rank

Tuesday, September 27

Warmup

Given that B is an invertible matrix, which one of these is not like the others: Col(A), Col(BA), Col(AB)?
ANSWER: Col(BA) is the odd one out. Example: $A = e_1$ and B swaps the rows of A.

If you have two sets A and B such that $A \subseteq B$ and $B \subseteq A$, what can you conclude about the sets?
ANSWER: You can conclude that $A = B$.

If you have a subspace $H \subset V$ such that $\dim(H) = \dim(V) = n$, what can you do to show that $H = V$?
You could show that $V \subset H$. If $H \subset V$ and $V \subset H$, then $V = H$.

Back to Bases

Let $\mathbb{P}_2$ be the set of polynomials of degree at most 2. Define $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C} = \{\frac{t^2-t}{2}, 1 - t^2, \frac{t^2+t}{2}\}$.

1. Express t^2 in terms of both $\mathcal{B}$ and $\mathcal{C}$.

   ANSWER: $t^2 = \mathbf{b}_3 = \mathbf{c}_1 + \mathbf{c}_3$.

2. If p is a polynomial such that $p(-1) = 1, p(0) = 0$ and $p(1) = 1$, express p in terms of both $\mathcal{B}$ and $\mathcal{C}$ (Use the fact that if $p = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$ then $p(x) = c_1\mathbf{b}_1(x) + c_2\mathbf{b}_2(x) + c_3\mathbf{b}_3(x)$ for all x, then solve the resulting linear system.)

   ANSWER: For the basis $\mathcal{B}$, we get the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

   which gives the solution $p = \mathbf{b}_3$, as before. (You can verify that t^2 takes on the values 1, 0, 1 at -1, 0 , 1, respectively.) For the basis $\mathbf{c}$ we get the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

   which gives the solution $p = \mathbf{c}_1 + \mathbf{c}_3$. The basis $\mathbf{c}$ is special in this case because each $\mathbf{c}_i$ takes on the value 1 at one of the locations $\{-1, 0, 1\}$ and 0 at the other two, making it a (theoretically) convenient basis for polynomial interpolation.

3. Let $\mathbb{P}_2$ be the set of polynomials of degree at most 2. Prove that if $T(p) = \{p(-1), p(0), p(1)\}$ then $T : \mathbb{P}_2 \to \mathbb{R}^3$ is a linear transformation. Is it an isomorphism? Justify your answer.

   ANSWER: To show that it is a linear transformation just verify that $T(p + q) = T(p) + T(q)$ and $T(c \cdot p) = cT(p)$. T is also an isomorphism. The most convenient way to show this is to note that

$\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ from before is a basis, and that $T(\mathbf{c}_1) = \mathbf{e}_1, T(\mathbf{c}_2) = \mathbf{e}_2, T(\mathbf{c}_3) = \mathbf{e}_3$. Since T takes a basis to another basis, it is an isomorphism.

4. Let $S \subseteq \mathbb{P}_2$ be the set of polynomials $p \in \mathbb{P}_2$ such that $p(3) = 0$. Find a basis for S and show that it is isomorphic to $\mathbb{R}^2$.

   ANSWER: The vectors $\{t - 3, t^2 - 9\}$ are linearly independent and both in S, so $\dim S \geq 2$. But $S \neq \mathbb{P}_2$, so $\dim S < 3$. Therefore $\dim S = 2$, and the already mentioned vectors are a basis.

   Since both S and $\mathbb{R}^2$ are 2-dimensional, they are isomorphic.

Rank

Let $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and take as given that they are row equivalent.
What is rank(A)? What is dim Nul(A)? Find basis for the column, row, and null spaces of A.
ANSWER: Since A and B are row equivalent they must have the same rank. Since B clearly has rank 2, so does A. By the Rank Theorem, the null space of A has dimension 4-2 = 2. The first two columns of A make a basis for the column space. The first two rows (since the row rank is also 2) make a basis for the row space of A. Use B and the two free variables in particular to get a basis for the null space of A, since A and B have the same null spaces.

If $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors, prove that $A = \mathbf{u}\mathbf{v}^T$ is a rank-1 matrix.
ANSWER: For any $\mathbf{x}$, $A\mathbf{x} = \mathbf{u}\mathbf{v}^T\mathbf{x} = \mathbf{u}(\mathbf{v}^T\mathbf{x}) = k\mathbf{u}$ for some k. Thus Col(A) = Span$\{\mathbf{u}\}$, which is a 1-dimensional space.