4.6,4.7,5.1: Coordinates, Eigenvalues

Thursday, September 29

Sanity Checks

How could you *efficiently* verify each of the following claims?

1. $\mathbf{z}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}$.
2. $\mathbf{y}$ is in the null space of A.
3. $[\mathbf{x}]_{\mathcal{B}} = \mathbf{y}$.
4. $[\mathbf{x}]_{\mathcal{B}} = [\mathbf{y}]_{\mathcal{C}}$.
5. $\mathbf{x}$ is an eigenvector of A.

Epic Pruf

Critique the following proofs:

Theorem 0.1 (Epic Theorem) *If* $\mathbf{v}_1 = \sin t, \mathbf{v}_2 = \sin 2t$, *and* $\mathbf{v}_3 = \sin t + \cos t \sin t$, *then* $\mathbf{v}_1, \mathbf{v}_2$, *and* $\mathbf{v}_3$ *are linearly indepednent.*

Proof: If the span of these vectors is V, define $T : V \to \mathbb{R}^3$ by $T(a_1 \sin t + a_2 \sin 2t + a_3 \cos t \sin t) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Then $[T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The columns of this matrix are linearly independent, and so form a basis for $\mathbb{R}^3$. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ was a basis for V.

Theorem 0.2 (Another Epic Theorem) *If* T *is a linear transformation and* $T(\mathbf{u})$ *and* $T(\mathbf{v})$ *are linearly dependent, then* $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly dependent.*

Proof: Prove the converse. If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent then there exist c_1 and c_2 such that $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$. But then $c_1 T(\mathbf{u}) + c_2 T(\mathbf{v}) = T(c_1\mathbf{u} + c_2\mathbf{v}) = T(\mathbf{0}) = \mathbf{0}$, so $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent.

Outer Products

Define $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$. Show that rank $\mathbf{u}\mathbf{v}^T \leq 1$. When is $\mathbf{u}\mathbf{v}^T$ rank zero?

Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases of a vector space V, and say $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.

1. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.
2. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$.
3. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

Eigenvectors!

If $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$, find the eigenvalues of A and find eigenvectors for those eigenvalues.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors such that $\mathbf{v}^T\mathbf{u} \neq 0$, then the matrix $\mathbf{u}\mathbf{v}^T$ has exactly one nonzero eigenvalue. What is it?

If V is the space of all infinitely differentiable functions and $D : V \to V$ is the derivative operator, find an eigenfunction with eigenvalue 2.