4.6,4.7,5.1: Coordinates, Eigenvalues

Thursday, September 29

Sanity Checks

How could you *efficiently* verify each of the following claims?

1. $\mathbf{z}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}$. Check that $A\mathbf{z} = \mathbf{b}$.
2. $\mathbf{y}$ is in the null space of A. Check that $A\mathbf{y} = \mathbf{0}$
3. $[\mathbf{x}]_{\mathcal{B}} = \mathbf{y}$. Check that $B\mathbf{y} = \mathbf{x}$
4. $[\mathbf{x}]_{\mathcal{B}} = [\mathbf{y}]_{\mathcal{C}}$. Check that $B^{-1}\mathbf{x} = C^{-1}\mathbf{y}$? That one isn't really "efficient".
5. $\mathbf{x}$ is an eigenvector of A. Check that $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$. Then λ is that multiple.

Epic Pruf

Critique the following proofs:

Theorem 0.1 (Epic Theorem) *If* $\mathbf{v}_1 = \sin t, \mathbf{v}_2 = \sin 2t$, *and* $\mathbf{v}_3 = \sin t + \cos t \sin t$, *then* $\mathbf{v}_1, \mathbf{v}_2$, *and* $\mathbf{v}_3$ *are linearly indepednent.*

Proof: If the span of these vectors is V, define $T : V \to \mathbb{R}^3$ by $T(a_1 \sin t + a_2 \sin 2t + a_3 \cos t \sin t) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$.

Then $[T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The columns of this matrix are linearly independent, and so form a basis for $\mathbb{R}^3$. Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ was a basis for V.

ANSWER: Note that the proof cannot be right because the three vectors are not linearly independent. In particular, $\sin 2t$ and $\sin t \cos t$ are not linearly independent, so T is not even a well-defined function! We would have $(0, 1, 0) = T(\sin 2t) = T(2 \sin t \cos t) = (0, 0, 2)$, which is a contradiction.

Theorem 0.2 (Another Epic Theorem) *If T is a linear transformation and $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent, then $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent.*

Proof: Prove the converse. If $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent then there exist c_1 and c_2 such that $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$. But then $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = T(c_1\mathbf{u} + c_2\mathbf{v}) = T(\mathbf{0}) = \mathbf{0}$, so $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent.

ANSWER: All of the logic in the answer is correct, but the converse is not equivalent to the original statement. (Note that the converse is true, but the original statement is not!) If the original statement had said *independent* both times instead of *dependent*, then what we proved would be the contrapositive and therefore equivalent.

Outer Products

Define $\mathbf{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$. Show that rank $\mathbf{u}\mathbf{v}^T \leq 1$. When is $\mathbf{u}\mathbf{v}^T$ rank zero?

You could work out the algebra and show that $\mathbf{u}\mathbf{v}^T$ row reduces to a matrix with one pivot. Simpler: $\mathbf{u}\mathbf{v}^T\mathbf{x} = \mathbf{u}(\mathbf{v}^T\mathbf{x}) = k\mathbf{u}$ where k is a scalar equal to $(\mathbf{v}^T\mathbf{x})$. Therefore the column space is spanned by $\mathbf{u}$, and so is 1-dimensional at most.

If either $\mathbf{u} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}$, then the matrix will be the zero matrix and therefore rank zero.

Change of Basis

Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases of a vector space V, and say $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.

1. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$: $\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$

2. Find $[\mathbf{x}]_{\mathcal{C}}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$.

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

3. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

$$\begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}^{-1}$$

Eigenvectors!

If $A = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}$, find the eigenvalues of A and find eigenvectors for those eigenvalues.
ANSWER: Since A is lower triangular its eigenvalues are just the diagonal elements: $\lambda_1 = 3, \lambda_2 = 1$. Then the eigenvectors are elements in the null spaces of $(A - 3I)$ and $(A - I)$, respectively. Two that would work are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Verify that $A\mathbf{v} = \lambda\mathbf{v}$ for each of these as a sanity check.

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors such that $\mathbf{v}^T\mathbf{u} \neq 0$, then the matrix $\mathbf{u}\mathbf{v}^T$ has exactly one nonzero eigenvalue. What is it?
Since $\mathbf{u}\mathbf{v}^T\mathbf{x} = (\mathbf{v}^T\mathbf{x})\mathbf{u}$, $\mathbf{x}$ must be parallel to $\mathbf{u}$ in order to be an eigenvector. So $\mathbf{x} = \mathbf{u}$ will do, in which case the eigenvalue is $\mathbf{v}^T\mathbf{u}$.

If V is the space of all infinitely differentiable functions and $D : V \to V$ is the derivative operator, find an eigenfunction with eigenvalue 2.
Do you remember separable differential equations from 1B?

$$\begin{aligned} Dy &= 2y \\ \frac{dy}{dx} &= 2y \\ \frac{1}{y}\,dy &= 2\,dx \\ \int \frac{1}{y}\,dy &= \int 2\,dx \\ \ln y &= 2x + c \\ y &= e^{2x+c} \\ y &= ke^{2x} \end{aligned}$$

Therefore any function of the form $f(x) = ke^{2x}$ is an eigenvector with eigenvalue 2. In general, any function of the form $f(x) = ke^{\lambda x}$ is an eigenfunction with eigenvalue λ.
More on this later.