4.3-4.4: Bases and Coordinate Systems

Thursday, September 22

Warmup

True/False:

1. If A is an $n \times n$ matrix and $\det(A) = 0$ then A has two identical rows/columns or a row/column of all zeros.

2. Let H be a subset of a vector space V. Suppose that $0 \in H$, $\mathbf{x}$ and $\mathbf{y}$ are vectors in H such that $\mathbf{x} + \mathbf{y} \in H$, and $c \in \mathbb{R}$ is a scalar such that $c\mathbf{x} \in H$. Then H is a subspace of V.

3. If a set of vectors in $\mathbb{R}^n$ is linearly independent and spans $\mathbb{R}^n$, then the set has exactly n vectors.

Define $\mathbf{u} = (1, 0), \mathbf{v} = (0, 1), \mathbf{w} = (1, 1)$, and suppose you want to get to the point $(5, 3)$. How do you get there if you can only travel parallel to $\mathbf{u}$ and $\mathbf{v}$? What if you can only travel parallel to $\mathbf{v}$ and $\mathbf{w}$?

If we define the basis $\mathcal{B} = \{\mathbf{v}, \mathbf{w}\}$, sketch $\mathbb{R}^2$ and the point $(5, 3)$ according to $\mathcal{B}$.

Bases

In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + z = 0$ (Hint: these vectors are solutions to some linear system).

If $A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix}$, find a basis for the null space of A.

Coordinate Systems

Consider three bases for $\mathbb{P}_2$: $\mathcal{B} = \{1, t, t^2\}, \mathcal{C} = \{1, t, 2t^2 - 1\}, \mathcal{D} = \{\frac{t^2-t}{2}, \frac{t^2+t}{2}, 1 - t^2\}$. Use coordinate vectors to verify that all three of these are bases.

Express the vector t^2 using each of these three bases.

If we want to find a polynomial $P(t) = a_0 + a_1 t + a_2 t^2$ such that $P(-1) = 4, P(0) = 2$, and $P(1) = 2$, which of the bases $\mathcal{B}$ or $\mathcal{D}$ is more convenient for finding $P(t)$? Why?

Isomorphisms

The set of vectors $V \subset \mathbb{R}^3$ such that $x - 3y + z = 0$ is not $\mathbb{R}^2$, but it is isomorphic to $\mathbb{R}^2$. Find an isomorphism $T : V \to \mathbb{R}^2$ (Hint: you only need to say how T acts on a basis of V).

Show that $\mathbb{R}$ and $\{0\}$ are not isomorphic. Show that $\mathbb{R}^2$ and $\mathbb{R}$ are not isomorphic.