4.3-4.4: Bases and Coordinate Systems

Thursday, September 22

Warmup

True/False:

1. If A is an $n \times n$ matrix and $\det(A) = 0$ then A has two identical rows/columns or a row/column of all zeros. FALSE: these are *sufficient* but not *necessary* to have a linear dependence between the columns.

2. Let H be a subset of a vector space V. Suppose that $0 \in H$, $\mathbf{x}$ and $\mathbf{y}$ are vectors in H such that $\mathbf{x} + \mathbf{y} \in H$, and $c \in \mathbb{R}$ is a scalar such that $c\mathbf{x} \in H$. Then H is a subspace of V. FALSE: this must be true for *all* pairs of vectors in H and *all* scalars.

3. If a set of vectors in $\mathbb{R}^n$ is linearly independent and spans $\mathbb{R}^n$, then the set has exactly n vectors. TRUE.

Define $\mathbf{u} = (1,0), \mathbf{v} = (0,1), \mathbf{w} = (1,1)$, and suppose you want to get to the point $(5,3)$. How do you get there if you can only travel parallel to $\mathbf{u}$ and $\mathbf{v}$? What if you can only travel parallel to $\mathbf{v}$ and $\mathbf{w}$?
$(5,3) = 5\mathbf{u} + 3\mathbf{v} = 5\mathbf{w} - 2\mathbf{v}$.

If we define the basis $\mathcal{B} = \{\mathbf{v}, \mathbf{w}\}$, sketch $\mathbb{R}^2$ and the point $(5,3)$ according to $\mathcal{B}$.
$(5,3)$ should be at the location $(-2,5)$ according to the transformed coordinate axes.

Bases

In the vector space of all real-valued functions, find a basis for the subspace spanned by $\{\sin t, \sin 2t, \sin t \cos t\}$.

Since $\sin 2t = 2 \sin t \cos t$ the vectors $\{\sin t, \sin 2t\}$ form a basis.
Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x - 3y + z = 0$ (Hint: these vectors are solutions to some linear system).
A basis could be $\{(3,1,0), (-1,0,1)\}$.

If $A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix}$, find a basis for the null space of A.

$$\begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 3 & -1 & -7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & -1 & -1 & -3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The null space is 1-dimensional and given by the equations $x_4 = 0$, x_3 is free, $x_2 + x_3 = 0$, and $x_1 = 2x_3$, so the null space is spanned by $\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

Multiply A with this vector and verify that it is zero as a sanity check.

Coordinate Systems

Consider three bases for $\mathbb{P}_2$: $\mathcal{B} = \{1, t, t^2\}, \mathcal{C} = \{1, t, 2t^2 - 1\}, \mathcal{D} = \{\frac{t^2-t}{2}, \frac{t^2+t}{2}, 1 - t^2\}$. Use coordinate vectors to verify that all three of these are bases.

If we use the transformation $a_0 + a_1 t + a_2 t^2 \mapsto (a_0, a_1, a_2)$, then the bases get transformed to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ \frac{-1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix},$$

all of which have nonzero determinant and therefore have linearly independent columns.

Express the vector t^2 using each of these three bases.

$t^2 = \mathbf{b}_1 = (\mathbf{c}_3 + \mathbf{c}_1)/2 = {}_1 + {}_2$.

If we want to find a polynomial $P(t) = a_0 + a_1 t + a_2 t^2$ such that $P(-1) = 4, P(0) = 2$, and $P(1) = 2$, which of the bases $\mathcal{B}$ or $\mathcal{D}$ is more convenient for finding $P(t)$? Why?

The basis $\mathcal{D}$ is more convenient because the coordinates are exactly the values of the polynomial at $1, -1$, and 0, respectively.

Isomorphisms

The set of vectors $V \subset \mathbb{R}^3$ such that $x - 3y + z = 0$ is not $\mathbb{R}^2$, but it is isomorphic to $\mathbb{R}^2$. Find an isomorphism $T : V \to \mathbb{R}^2$ (Hint: you only need to say how T acts on a basis of V).

Take $T(3, 1, 0) = (1, 0)$ and $T(1, 0, -1) = (0, 1)$. We could more easily specify an isomorphism in the other direction: if $S(1, 0) = (3, 1, 0)$ and $S(0, 1) = (1, 0, -1)$, then S has the representation $\begin{bmatrix} 3 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ and

$$TS = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Show that $\mathbb{R}$ and $\{0\}$ are not isomorphic. Show that $\mathbb{R}^2$ and $\mathbb{R}$ are not isomorphic.
Easy answer: they have different dimensions.
Answer from first principles: Let $T : \mathbb{R} \to \{0\}$ be a linear transformation. Then $T(1) = T(0) = 0$, so T cannot be onto. Thus there is no linear transformation that is 1-1 and onto.
Similarly, let $T\mathbb{R}^2 \to \mathbb{R}$ be a linear transformation. Let $T(1, 0) = a, T(0, 1) = b$. Then either $a = 0$, in which case T is not 1-1, or $a \neq 0$, in which case $T(-b/a, 1) = (-b/a) \cdot a + b = 0$, so again T is not 1-1. There is therefore no 1-1 linear transformation, and so the spaces are not isomorphic.