2.3-2.5: Matrix Algebra

Tuesday, September 13

Logic Warmup

State the *contrapositive* (i.e. "if (not B) then (not A)") of each of the following:

1. If mn is odd then m is odd and n is odd. **If m is even or n is even then mn is also even.**
2. If a number n is divisible by 2 and 3 then it is divisible by 6. **If n is not divisible by 6 then it is not divisible by 2 or not divisible by 3.**
3. If a function f is not one-to-one then for every function g, $g \circ f$ is also not one-to-one. **If there exists some g such that $g \circ f$ is one-to-one, then f is also one-to-one.**

Block Matrices

Evaluate each of the following matrix products:

1. $$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = 26.$$
2. $$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \\ 9 & 12 & 15 \end{bmatrix}$$

Prove the following (assuming all relevant matrices are $n \times n$). For which statements is it simpler to prove the contrapositive?

1. If $A\mathbf{x} = \mathbf{0}$ has a unique solution then A is one-to-one.

   The Inverse Matrix theorem implies this immediately. But if we want to prove it without the IMT, try the contrapositive: suppose that A is not 1-1, then there exist $v \neq w$ with $A(v) = A(w)$. But then $v - w \neq 0$ and $A(v - w) = 0$, so $A\mathbf{x} = \mathbf{0}$ does not have a unique solution.

2. A is invertible if and only if A^T is invertible.

   Direct: If A is invertible, then there exists B such that $BA = I$. But then $A^T B^T = (BA)^T = I^T = I$, so B^T is an inverse for A^T.

3. If A is not invertible then BA is also not invertible.

   Several different methods: if A is not invertible then $A\mathbf{x} = \mathbf{0}$ has some non-zero solution $\mathbf{v}$. But then $BA\mathbf{v} = B(A\mathbf{v}) = B\mathbf{0} = \mathbf{0}$, so BA has a non-trivial solution to the homogeneous equation and therefore is also not invertible.

4. If A is onto then A^2 is also onto.

   Let $\mathbf{b}$ be arbitrary. Then there is some $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$, but there is also some $\mathbf{w}$ such that $A\mathbf{w} = \mathbf{x}$. Then $A^2\mathbf{w} = A(A\mathbf{w}) = A\mathbf{x} = \mathbf{b}$, so A^2 is onto.

Remainder omitted because it was not relevant to the syllabus. Practice determinants instead!