4.1-4.2: Vector Spaces

Thursday, September 15

0.1 Matrix Patterns

Define $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Find DA, AD, PA, AP^T, and PAP^T.

ANSWER: DA scales the rows of A, AD scales the columns. PA swaps the rows of A, AP^T swaps the corresponding columns, and PAP^T does both.

Calculate the determinants of the following matrices

1. $\begin{bmatrix} 1 & 3 & -6 & \pi^\pi & 1 \\ 2 & \sqrt{2} & 77 & 0 & 2 \\ 3 & -3 & .3 & .03 & 3 \\ 4 & 10^{100} & \sin(12) & 8 & 4 \\ 5 & 0 & -5 & 0 & 5 \end{bmatrix}$: 0 because two columns are the same.

2. $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{bmatrix}$: 6, I think? It's either 6 or -6, but use cofactor expansion.

3. $\begin{bmatrix} -4 & 0 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 1 & 0 & -1 & -1 \\ 2 & 6 & 0 & 0 \end{bmatrix}$: lots of ways to reduce this, but I'll try by adding 4 times column 3 to column 1 as a start...

$$
\begin{aligned}
Det \begin{bmatrix} -4 & 0 & 1 & 0 \\ 0 & 3 & 3 & 2 \\ 1 & 0 & -1 & -1 \\ 2 & 6 & 0 & 0 \end{bmatrix} &= Det \begin{bmatrix} 0 & 0 & 1 & 0 \\ 12 & 3 & 3 & 2 \\ -3 & 0 & -1 & -1 \\ 2 & 6 & 0 & 0 \end{bmatrix} \\
&= Det \begin{bmatrix} 12 & 3 & 2 \\ -3 & 0 & -1 \\ 2 & 6 & 0 \end{bmatrix} \\
&= Det \begin{bmatrix} 6 & 3 & 0 \\ -3 & 0 & -3 \\ 2 & 6 & 0 \end{bmatrix} \\
&= 3 \cdot Det \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix} \\
&= 90.
\end{aligned}
$$

Subspaces

Describe, geometrically, *all* subspaces of $\mathbb{R}^3$.
All lines and planes that pass through the origin, but also $\{(0,0,0)\}$ and $\mathbb{R}^3$.

Which of the following are subspaces of the space of 2×2 matrices? Justify your answers:

1. $\{A : A = -A^T\}$: Yes. $A + B = -(A+B)^T$ and $cA = -cA^T$
2. $\{A : A^2 = I\}$: No: $I^2 = I$ but $(I+I)^2 \neq I$.

Derivative

Let V be the set of all infinitely differentiable functions on $\mathbb{R}$, and let $D : V \to V$ be the derivative operator (i.e. $D(f) = f'$).

1. Is D a linear tranformation? Justify your answer.

   Yup! $(cf + g)' = cf' + g'$ for any $c \in \mathbb{R}, f, g \in V$.

2. Which of the following are subspaces?

   (a) $\{f : D(f) = 0\}$: yes

   (b) $\{f : D(f) = 1\}$: no

   (c) $\{f : D(f) = ax + b, a, b, \in \mathbb{R}\}$: yes

3. What is the kernel of D? All constant functions.

4. Let $\mathbb{P}_2$ be the set of degree 2 polynomials, and represent the polynomial $a_2x^2 + a_1x + a_0$ $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$. What is the matrix representation of D?

$$D = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

5. What would this look like if we tried it for the space of *all* polynomials?

   Infinitely large matrix with $1, 2, 3, \ldots$ on the first superdiagonal? Does this even make sense?

If $A = \begin{bmatrix} 1 & 3 & -4 & -3 & 1 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, what are Col(A) and Nul(A)?

$$Col(A) = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

$$Nul(A) = \left\{ \begin{bmatrix} -5u + 6v - w \\ 3u - v \\ u \\ v \\ w \end{bmatrix} : u, v, w \in \mathbb{R} \right\} = \text{Span}\left(\begin{bmatrix} -5 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

If A and B are matrices, what is the relation between Col(A) and Col(AB)? What about Nul(B) and Nul(AB)?
$Col(AB) \subset Col(A)$ and $Nul(B) \subset Nul(AB)$. All of these sets are subspaces.