1.9,2.1: Linear Transformations, Matrix Algebra

Tuesday, September 6

More Logic

The *negation* of a statment is true if and only if that statement is false. For example, the negation of "All ravens are black" is "There exists a raven that is not black." Negate the following statements:

1. All roses are either red or white: There exists a rose that is not red and not white.
2. If a bird is black, then that bird is a raven: There is a bird that is black but not a raven.
3. There exist animals that have wings but cannot fly: Every animal with wings can fly (For every animal, if that animal has wings then it can fly).
4. If x is positive then $x^2 - 3x + 1$ is also positive: There is a positive x such that $x^2 - 3x + 1 \leq 0$.
5. If $AB = 0$ then $A = 0$ or $B = 0$: There exist $A, B \neq 0$ such that $AB = 0$.

Bonus warmup question: what is $\sum_{i=1}^{2} \sum_{j=1}^{2} 2^{i-j}$?

$$\sum_{i=1}^{2} \sum_{j=1}^{2} 2^{i-j} = 2^{1-1} + 2^{1-2} + 2^{2-1} + 2^{2-2} = 9/2.$$

Find negations for the following statements. Decide whether the statements are true or false and justify your answers.

1. If T is a linear transformation and T is one-to-one, then T is onto.

   Negation: There is a linear transformation T that is one-to-one but not onto.

   The original statement is false: take $T(x) = (x, 0)$ as a function from $\mathbb{R}$ to $\mathbb{R}^2$.

2. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is a linear transformation.

   Negation (cheap version): f is not a linear transformation.

   More precisely, a linear transformation must satisfy *both* of the following properties:

   (a) For all x and y, $T(x + y) = T(x) + T(y)$
   (b) For all x and scalars c, $T(cx) = c \cdot T(x)$

   f is therefore not linear if there exist x and y such that the first property is false **or** if there exist x and c such that the second property is false. As it turns out, both are false for $f(x) = x^2$:

   (a) $f(3 + 2) = 25$, but $f(3) + f(2) = 9 + 4 = 13$.
   (b) $f(-1 \cdot 5) = 25$, but $-1 \cdot f(5) = -25$.

3. The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = e^x - 3$ is onto.

   Cheap negation: g is not onto.

   Unpacked: g is onto if for **every** y there is an x such that $g(x) = y$, so g is **not** onto if there exists a y such that $g(x)$ is never equal to y.

   To show that g is not onto, let $y = -3$. Then for any x, $g(x) = e^x - 3 > 0 - 3 = -3$, so $g(x) \neq -3$ for any x.

State the negations of the following statements, and find counterexamples with 2×2 matrices: "For all matrices A and B…"

- $AB = BA$. (Hint: try a shear and a reflection or rotation)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- If $A \neq 0$ and $AB = AC$, then $B = C$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}.$$

- If $AB = 0$ then $A = 0$ or $B = 0$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$$

Linear Transformations

If $f(x) = mx + b$, for what values of m and b is f a linear transformation? When it is linear, express its standard matrix representation in terms of m and b.

If f is linear, then $f(x+y) = f(x)+f(y)$ for any x and y. This implies that $m(x+y)+b = (mx+b)+(my+b)$, which in turn means that $b = 0$. So b must be zero, but any value of m will do. The transformation then has the matrix representation $\begin{bmatrix} m \end{bmatrix}$

If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, illustrate the effect of A on the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. Find A^2, A^3, and A^4, and describe the associate linear transformations.

$$A\mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A\mathbf{e}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

A has the effect of rotating the plane 90 degrees counterclockwise. A^2 is rotation 180 degrees, A^3 rotates 270 degrees, and A^4 rotates 360 degrees, which is the same as not rotating at all! A^4 should therefore be the identity matrix.

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, what is A^n? Describe the geometric effect of applying A to a vector repeatedly. Find a matrix B such that $AB = I$.

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

A has a shearing effect, and A^n just shears by even more. A^{-1} is a matrix that shears in the opposite direction.

If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, find all possible combinations of products of A and B and illustrate their effects on the letter "R".

There should only be 8 possible such products, and they correspond to all possible horizontal/vertical reflections and 90 degree rotations of the plane. These 8 products are characterized by the relations

$$
\begin{aligned}
A^4 &= I \\
B^2 &= I \\
AB &= BA^3
\end{aligned}
$$