2.1-2.3: Matrices and Inverses

Thursday, September 6

Proofs

If Trevor gets stuck in traffic he will be late to work. If he is late to work he will be fired. He will not get stuck in traffic if *and only if* he takes the shortcut.
Which of the following conclusions are logically valid? Prove the ones that are.

1. If Trevor does not take the shortcut then he will be fired.
2. If Trevor takes the shortcut the he will not be fired.
3. If Trevor is not late to work then he took the shortcut.

Critique the following proofs:

Theorem: If $x^2 + 1 = 5$ then $x = 2$.

$$\begin{aligned} x &= 2 \\ 2x &= 4 \\ 2x - 2 - x &= 4 - 2 - x \\ x - 2 &= 2 - x \\ (x-2)^2 &= (2-x)^2 \\ x^2 - 4x + 4 &= x^2 - 4x + 4 \\ 0 &= 0 \end{aligned}$$

Theorem: If $Ap = b$, $Av = 0$, and $w = p + v$, then $Aw = b$.

$$\begin{aligned} Aw &= b \\ A(p+v) &= b \\ Ap + Av &= b \\ b + 0 &= b \\ 0 &= 0 \end{aligned}$$

If $X = \{1,2,3,4\}, Y = \{5,6,7\}$, is there a function $g : X \to X$ that is one-to-one but not onto? Onto but not one-to-one? What about a function from X to Y? Y to X?
Answer the same for linear transformations from and to $\mathbb{R}^3$ and $\mathbb{R}^4$.

Prove: If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent but $T(\mathbf{u}), T(\mathbf{v})$ are linearly dependent, then $T\mathbf{x} = 0$ has a nontrivial solution.
Prove: If for every $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution, then the function $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

Matrix Inverses

Let $A = \begin{bmatrix} 1 & 5 \\ -2 & -7 \end{bmatrix}$. Find a sequence of elementary matrices E_1, E_2, E_3 that put A in reduced echelon form. What is $E_3E_2E_1$? How does this compare with A^{-1}?

If $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, describe the effect of its associated linear transformation. Find A^{-1} and describe its associated linear transformation.

Say that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose that $a \neq 0$ but $ad - bc = 0$. What happens when you put A in echelon form?

If $S : \mathbb{R}^4 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^4$ are linear transformations then so are $T \circ S$ and $S \circ T$. Can $T \circ S$ have an inverse? What about $S \circ T$?