2.1-2.3: Matrices and Inverses

Thursday, September 6

Proofs

If Trevor gets stuck in traffic he will be late to work. If he is late to work he will be fired. He will not get stuck in traffic if *and only if* he takes the shortcut.
Which of the following conclusions are logically valid? Prove the ones that are.

1. If Trevor does not take the shortcut then he will be fired. VALID
2. If Trevor takes the shortcut the he will not be fired. NOT VALID
3. If Trevor is not late to work then he took the shortcut. VALID

Critique the following proofs:
Theorem: If $x^2 + 1 = 5$ then $x = 2$.

$$\begin{aligned} x &= 2 \\ 2x &= 4 \\ 2x - 2 - x &= 4 - 2 - x \\ x - 2 &= 2 - x \\ (x-2)^2 &= (2-x)^2 \\ x^2 - 4x + 4 &= x^2 - 4x + 4 \\ 0 &= 0 \end{aligned}$$

This proof is very wrong. Two warning signs: first, it did not ever use the given information that $x^2+1 = 5$. Second, the stated theorem is false! The basic problem is that if $x = y$ then $x^2 = y^2$ but $x^2 = y^2$ does not imply that $x = y$. Thus the step from line 4 to line 5 is not "reversible", and so starting from the conclusion and working to a true statement does not imply that our desired conclusion ($x = 2$) is true.
A correct presentation of a slightly modified theorem would be as follows:

$$\begin{aligned} x^2 + 1 &= 5 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

Theorem: If $Ap = b$, $Av = 0$, and $w = p + v$, then $Aw = b$.

$$\begin{aligned} Aw &= b \\ A(p+v) &= b \\ Ap + Av &= b \\ b + 0 &= b \\ 0 &= 0 \end{aligned}$$

The presentation is not great because it starts from the conclusion and works to a true statement, but all of the equalities in the proof are equivalent to each other so the logic basically works. A better presentation would be:

$$
\begin{aligned}
w &= p + v \\
Aw &= A(p + v) \\
Aw &= Ap + Av \\
Aw &= b + 0 \\
Aw &= b,
\end{aligned}
$$

where if you want to be extra clear you can cite your reasoning for making each step. If you want to give as short an answer as possible then the following would do:

$$Aw = A(p + v) = Ap + Av = b + 0 = b.$$

If $X = \{1, 2, 3, 4\}, Y = \{5, 6, 7\}$, is there a function $g : X \to X$ that is one-to-one but not onto? Onto but not one-to-one? What about a function from X to Y? Y to X?
Answer the same for linear transformations from and to $\mathbb{R}^3$ and $\mathbb{R}^4$.

The answer for functions on sets of size m and n and *linear* transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$ are the same!

	$\mathbb{R}^4 \to \mathbb{R}^4$	$\mathbb{R}^4 \to \mathbb{R}^3$	$\mathbb{R}^3 \to \mathbb{R}^4$
1-1 but not onto?	NO	NO	YES
onto but not 1-1?	NO	YES	NO

The key idea here is that even though $\mathbb{R}$ contains infinitely many points, linear transformations only carry a finite amount of "information": in particular, a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ can be described entirely by an $m \times n$ matrix. You couldn't possibly describe arbitrary functions so succinctly!
The reason for this is the properties that all linear transformations follow. If we know $T(x)$ and $T(y)$, we also automatically know $T(3x), T(x + y), T(x/2 - \pi y), \ldots$ More precisely, we know $T(z)$ for every $z \in \text{Span}(x, y)$. So if we can span $\mathbb{R}^m$ with a set of m vectors, we only need to know what T does to those m vectors to know what T does to any point in $\mathbb{R}^m$.

Prove: If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent but $T(\mathbf{u}), T(\mathbf{v})$ are linearly dependent, then $T\mathbf{x} = 0$ has a nontrivial solution.

If $T(\mathbf{u})$ and $T(\mathbf{v})$ are linearly dependent then there exist nonzero c_1 and c_2 such that $c_1T(\mathbf{u}) + c_2T(\mathbf{v}) = 0$. But then $0 = c_1T(\mathbf{u})+c_2T(\mathbf{v}) = T(c_1\mathbf{u}+c_2\mathbf{v})$, and we know that $c_1\mathbf{u}+c_2\mathbf{v} \neq 0$ since $\mathbf{u}$ and $\mathbf{v}$ are independent.

Prove: If for every $\mathbf{b}$ the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution, then the function $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

In matrix terms: if $A\mathbf{x} = \mathbf{b}$ has at most one solution then the reduced echelon form of A has no free variables. This also means that $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
This is also a little informal, so if we want to be rigorous the best method is to use *proof by contraposition*: to prove "if P, then Q" we will prove the equivalent statement "if NOT Q, then NOT P". Our goal is therefore to prove that if $\mathbf{x} \mapsto A\mathbf{x}$ is NOT one-to-one, then there is a $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has many solutions.
Proof: if the map is not one-to-one, then there exist $\mathbf{v}$ and $\mathbf{w}$ such that $A\mathbf{v} = A\mathbf{w}$. But then $\mathbf{v} - \mathbf{w} = 0$ and $A(\mathbf{v} - \mathbf{w}) = A\mathbf{v} - A\mathbf{w} = 0$, implying that $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution (and therefore infinitely many solutions).

Matrix Inverses

Let $A = \begin{bmatrix} 1 & 5 \\ -2 & -7 \end{bmatrix}$. Find a sequence of elementary matrices E_1, E_2, E_3 that put A in reduced echelon form. What is $E_3E_2E_1$? How does this compare with A^{-1}?

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -2 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then $E_3E_2E_1 = \begin{bmatrix} -7/3 & -5/3 \\ 2/3 & 1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -7 & -5 \\ 2 & 1 \end{bmatrix}$, which is just A^{-1}.
In particular, if we want to solve $A\mathbf{x} = \mathbf{b}$ then "row reduction" is the same as multiplying by E_1, E_2, and E_3 in turn. Additionally, "reduced echelon form" for an invertible matrix is the identity matrix.

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ E_3E_2E_1A\mathbf{x} &= E_3E_2E_1\mathbf{b} \\ I\mathbf{x} &= E_3E_2E_1\mathbf{b} \\ \mathbf{x} &= E_3E_2E_1\mathbf{b} \end{aligned}$$

An easy way to find E from the desired row operations: since $E \cdot I = E$, think of how applying E would effect the identity matrix. For example, if the desired effect of E is "add 2 times row 1 to row 2", applying that row operation to the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ would give $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, which is therefore E.

If $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, describe the effect of its associated linear transformation. Find A^{-1} and describe its associated linear transformation.
A corresponds to 45 degree counterclockwise rotation.
$A^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ corresponds to 45 degree clockwise rotation.

Say that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose that $a \neq 0$ but $ad - bc = 0$. What happens when you put A in echelon form?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ ac & ad \end{bmatrix}$$
$$\sim \begin{bmatrix} a & b \\ 0 & ad-bc \end{bmatrix}$$
$$\sim \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

Therefore, if $ad-bc=0$ then the reduced echelon form of A has only a single pivot and so A is not invertible. Otherwise,

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $S : \mathbb{R}^4 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^4$ are linear transformations then so are $T \circ S$ and $S \circ T$. Can $T \circ S$ have an inverse? What about $S \circ T$?

Without proof, $S \circ T : \mathbb{R}^3 \to \mathbb{R}^3$ can have an inverse but $T \circ S : \mathbb{R}^4 \to \mathbb{R}^4$ cannot. Loosely, the idea for $T \circ S$ is that we have to compress $\mathbb{R}^4$ to take it down to $\mathbb{R}^3$, but then no linear transformation is capable of "recovering" that information. A little more formally, since T cannot be onto it follows that $T \circ S$ also cannot be onto.

On the other hand, it is relatively simple to embed $\mathbb{R}^3$ in $\mathbb{R}^4$ and then take it back again. (Think taking 3D space to 3D space at time zero in 4D spacetime, then taking it back to 3D space). As a matrix example, we can take

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, ST = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3,$$

so ST is certainly invertible.

Moral of the story: A short fat matrix times a tall skinny one could give the identity matrix (or an invertible matrix in general) but a tall skinny one times a short fat one cannot.