1.3-1.4: Vector Equations and $Ax = b$

Tuesday, August 30

Functions

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = 2x$. What is $\{f(x) : x \in \mathbb{R}\}$? Does $f(x)$ have an inverse function? If so, is its inverse defined on all of $\mathbb{R}$?
The image/range of f ($\{f(x) : x \in \mathbb{R}\}$) is all of $\mathbb{R}$. $f^{-1}(x) = x/2$ is an inverse function, and it is defined on all of $\mathbb{R}$.

Answer the same questions as above for $g(x) = x^2$ and $h(x) = e^x$.
The image/range of g is $[0, \infty)$. g does not have an inverse because it is not one-to-one: for example $g(-2) = g(2) = 4$, so an inverse function would have to satisfy both $g^{-1}(4) = -2$ and $g^{-1}(4) = 2$, which is impossible.
If we restrict the *domain* of g to be the interval $[0, \infty)$, then g is one-to-one and $\sqrt{x}$ is a well-defined inverse.
The image of h is $(0, \infty)$. h is one-to-one and $\log(x)$ is an inverse function for h, but it is only defined for real numbers on the interval $(0, \infty)$.

Let's make a secret code with the encryption function E('a') = 'm', E('b') = 'm',..., E('z') = 'm'. Use it to encrypt the message "hello world". Is this a useful secret code?
The encoded message is "mmmmm mmmmm". This is not very useful since it is impossible to decode. Since the encryption function is not one-to-one, information is lost during the encryption process.

If possible, find functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \{a, b, c\} \to \{1, 2, 3\}$ that are...

- 1-1 but not onto: e^x, NONE
- onto but not 1-1: $x^3 - x$, NONE
- both: $2x$, $g(a) = 1, g(b) = 2, g(c) = 3$
- neither: $f(x) = 0$, $g(a) = g(b) = g(c) = 1$

The lesson: for finite sets, there is a stricter relation between the size of the set and whether the function is one-to-one or onto than for infinite sets. We will see that similar patterns hold when dealing with linear functions.

Vector Equations

Define $\mathbf{u} := \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v} := \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{w} := \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ for the rest of this section. Sketch the following:

- $\mathbf{u} + \mathbf{v}$
- $\mathbf{v} + \mathbf{u}$
- $\frac{1}{2}\mathbf{w} - 2\mathbf{u}$
- $\mathbf{w} + 2\mathbf{v}$

- Span($\mathbf{v}, \mathbf{w}$): just a straight line parallel to $\mathbf{v}$
- Span($\mathbf{u}, \mathbf{v}$): all of $\mathbb{R}^2$

If we define $\mathbf{b} := \begin{bmatrix} 3 \\ 5 \end{bmatrix}$, is $\mathbf{b}$ in Span($\mathbf{u}, \mathbf{v}, \mathbf{w}$)? What about Span($\mathbf{v}, \mathbf{w}$)?
It is definitely true that $\mathbf{b} \in \text{Span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ because the span of these three vectors is all of $\mathbb{R}^2$, so *every* vector in $\mathbb{R}^2$ is in the span.
$\mathbf{b}$ is not in the span of $\{\mathbf{v}, \mathbf{w}\}$ since (geometrically) the vector $\mathbf{b}$ does not sit on the line defined by the span (as sketched in the previous exercise). You can also reduce the augmented system $\begin{bmatrix} \mathbf{v} & \mathbf{w} & \mathbf{b} \end{bmatrix}$ to echelon form to verify this fact.

Find a vector $\mathbf{y}$ such that the system $c_1\mathbf{v} + c_2\mathbf{w} = \mathbf{y}$ has infinitely many solutions.
Any multiple of $\mathbf{v}$ (which is also a multiple of $\mathbf{w}$), and *only* multiples of $\mathbf{v}$, will have infinitely many solutions.
To prove this first part: you can check that $\mathbf{w} = -2\mathbf{v}$, so if $\mathbf{y} = \alpha\mathbf{v}$, it follows that for any $t \in \mathbb{R}$, we will have $t\mathbf{w} + (2t + \alpha)\mathbf{v} = -2t\mathbf{v} + 2\mathbf{v} + \alpha\mathbf{v} = \alpha\mathbf{v} = \mathbf{y}$.
You can also reduce the augmented system $\begin{bmatrix} \mathbf{v} & \mathbf{w} & \mathbf{y} \end{bmatrix}$ to echelon form and verify that there is a free variable.

A robot begins at the point $(0, 0, 0)$ and is capable of moving in the directions $\pm(1, 1, 1)$ and $\pm(-1, 3, 0)$. Find a point in space that the robot cannot reach.
To find **all** points that the robot cannot reach, we can try to see whether an arbitrary point $[x, y, z]$ is in the span of the vectors $(1, 1, 1)$ and $(-1, 3, 0)$:

$$\begin{aligned} \begin{bmatrix} 1 & -1 & x \\ 1 & 3 & y \\ 1 & 0 & z \end{bmatrix} &\sim \begin{bmatrix} 1 & -1 & x \\ 0 & 4 & y - x \\ 0 & 1 & z - x \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & x \\ 0 & 1 & z - x \\ 0 & 4 & y - x \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -1 & x \\ 0 & 1 & z - x \\ 0 & 0 & (y - x) - 4(z - x) \end{bmatrix}. \end{aligned}$$

The system is therefore consistent if and only if $y + 3x - 4z = 0$, and any other vector will be out of the robot's reach. Thus, for example, the robot cannot reach the point $(1, 0, 0)$.
To be honest, pretty much any "random" guess would have worked. This is because the span of the robot's direction vectors forms a plane, but planes take up very little of 3-dimensional space (i.e. they have zero volume).

Matrix Equations

If we define $A := \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$, write $\mathbf{v} + \mathbf{u}$, $\frac{1}{2}\mathbf{w} - 2\mathbf{u}$, and $\mathbf{w} + 2\mathbf{v}$ as matrix-vector products.
$A \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} -2 \\ 0 \\ -1/2 \end{bmatrix}$, and $A \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ are the correct matrix-vector products.

Recall that if A is an m-by-n matrix then

- $A(\mathbf{v} + \mathbf{u}) = A\mathbf{v} + A\mathbf{u}$
- $A(c\mathbf{v}) = c \cdot A\mathbf{v}$

for any $\mathbf{v}, \mathbf{u} \in \mathbb{R}^n$ and any $c \in \mathbb{R}$. Using these two facts, prove that if $A\mathbf{v}_1 = \mathbf{w}_1$ and $A\mathbf{v}_2 = \mathbf{w}_2$ then the system $A\mathbf{x} = 3\mathbf{w}_1 - \mathbf{w}_2$ is consistent.
Let $\mathbf{x} = 3\mathbf{v}_1 - \mathbf{v}_2$. Then $A\mathbf{x} = A(3\mathbf{v}_1 - \mathbf{v}_2) = A(3\mathbf{v}_1) + A(-\mathbf{v}_2) = 3A\mathbf{v}_1 - A\mathbf{v}_2 = 3\mathbf{w}_1 - \mathbf{w}_2$. This shows that the system is consistent. Note: this was covered in more depth in Thursday's discussion.