1.5,1.7-1.9: Linear Independence, Linear Transformations

Thursday, September 1

Logic Warmup

Which of the following are logically equivalent to "All ravens are black"?

1. If x is a raven, then x is black.
2. If x is black, then x is a raven.
3. If x is not a raven, then x is not black.
4. If x is not black, then x is not a raven.

All humans are mortal, for that is the way of life. All mortals are afraid of bees, for their sting is deadly. If Odin is not mortal, which of the following *must* be true?

1. Odin is not human.
2. Odin is not afraid of bees.

Existence Proofs

Rephrase each of the following statements in the form "There exist(s) X such that Y."

1. The equation $3x + 5 = 14$ has a solution.
2. $f(x) = x^2$ for $x \in \mathbb{R}$, then $9 \in \text{Range}(f)$.
3. The system $A\mathbf{x} = \mathbf{b}$ is consistent.
4. $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$.
5. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set.
6. The function $h : \mathbb{R} \to \mathbb{R}$ is *not* one-to-one.

Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = 0$. Show that $A(c\mathbf{v} + d\mathbf{w}) = \mathbf{0}$ for all $c, d \in \mathbb{R}$.

Prove or give a counterexample: if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors such that $\mathbf{v}_3 \notin \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Suppose that T, a linear transformation, is *not* one-to-one. Show that $T\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

Matching Exercise

The reduced echelon forms of four matrices are given below. If T is the linear transformation associated with a matrix, match the following statements with each other and with the appropriate matrix/matrices:

$$\begin{bmatrix} \bullet & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & \bullet \end{bmatrix} \quad \begin{bmatrix} \bullet & 0 & 0 & * \\ 0 & \bullet & 0 & * \\ 0 & 0 & \bullet & * \end{bmatrix} \quad \begin{bmatrix} \bullet & 0 & * \\ 0 & \bullet & * \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \bullet & 0 \\ 0 & \bullet \\ 0 & 0 \end{bmatrix}$$

1. The columns of A span $\mathbb{R}^3$.
2. T is one-to-one.
3. T is onto.
4. There is no $\mathbf{x}$ such that $A\mathbf{x} = 0$.
5. T is neither one-to-one nor onto.
6. The columns of A are linearly independent.
7. $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.
8. Any two columns of A are linearly dependent.
9. $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^3$.
10. There exists a $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.
11. One of the columns of A can be deleted and A will still span $\mathbb{R}^3$.
12. There exists some $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.
13. There exists $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.
14. Every $\mathbf{w} \in \mathbb{R}^3$ is a linear combination of the columns of A.

Miscellany

Let $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$. Sketch the set of all $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ and the set of all $\mathbf{x}$ such that $A\mathbf{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

Draw a capital "R" in the first quadrant of $\mathbb{R}^2$. What happens to the image if you apply the transformations $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, or $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$?