1.5,1.7-1.9: Linear Independence, Linear Transformations

Thursday, September 1

Logic Warmup

Which of the following are logically equivalent to "All ravens are black"?

1. If x is a raven, then x is black: YES
2. If x is black, then x is a raven: NO
3. If x is not a raven, then x is not black: NO
4. If x is not black, then x is not a raven: YES

In general, the first and fourth statements are always equivalent to each other and the second and third statements are equivalent to each other, but these two pairs of statements are not equivalent.

All humans are mortal, for that is the way of life. All mortals are afraid of bees, for their sting is deadly. If Odin is not mortal, which of the following *must* be true?

1. Odin is not human. YES
2. Odin is not afraid of bees. NO: perhaps everyone is afraid of bees because their stings hurt.

Existence Proofs

Rephrase each of the following statements in the form "There exist(s) X such that Y."

1. The equation $3x + 5 = 14$ has a solution. There exists x such that $3x + 5 = 14$
2. $f(x) = x^2$ for $x \in \mathbb{R}$, then $9 \in \text{Range}(f)$. There exists $x \in \mathbb{R}$ such that $x^2 = 9$
3. The system $A\mathbf{x} = \mathbf{b}$ is consistent. There exists $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$
4. $\mathbf{w} \in \text{Span}(\mathbf{u}, \mathbf{v})$. There exist $c_1, c_2 \in \mathbb{R}$ such that $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}$.
5. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set. There exists $c_1, c_2, c_3 \in \mathbb{R}$, *not* all equal to zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$.
6. The function $h : \mathbb{R} \to \mathbb{R}$ is *not* one-to-one. There exist $x, y \in \mathbb{R}$ such that $x \neq y$ but $h(x) = h(y)$.

Suppose that $\mathbf{v}$ and $\mathbf{w}$ are vectors such that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = 0$. Show that $A(c\mathbf{v} + d\mathbf{w}) = \mathbf{0}$ for all $c, d \in \mathbb{R}$.
For any $c, d \in \mathbb{R}$, $A(c\mathbf{v} + d\mathbf{w}) = A(c\mathbf{v}) + A(d\mathbf{w}) = cA\mathbf{v} + dA\mathbf{w} = c\mathbf{0} + d\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

Prove or give a counterexample: if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are vectors such that $\mathbf{v}_3 \notin \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$, then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
Counterexample: $\mathbf{v}_1 = \mathbf{v}_2 = (1, 0)$ but $\mathbf{v}_3 = (0, 1)$. Then $\mathbf{v}_3 \notin \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$ but the three vectors are linearly dependent because $\mathbf{v}_1 - \mathbf{v}_2 = 0$ ($c_1 = 1, c_2 = -1, c_3 = 0$).

Suppose that T, a linear transformation, is *not* one-to-one. Show that $T\mathbf{x} = \mathbf{0}$ has a nontrivial solution.
Since T is not one-to-one, there exists $\mathbf{v}$ and $\mathbf{w}$ such that $\mathbf{v} \neq \mathbf{w}$ but $T(\mathbf{v}) = T(\mathbf{w})$. Then $\mathbf{v} - \mathbf{w} \neq \mathbf{0}$ but $T(\mathbf{v} - \mathbf{w}) = T(\mathbf{v}) - T(\mathbf{w}) = \mathbf{0}$

Matching Exercise

The reduced echelon forms of four matrices are given below. If T is the linear transformation associated with a matrix, match the following statements with each other and with the appropriate matrix/matrices:

$$1: \begin{bmatrix} \bullet & 0 & 0 \\ 0 & \bullet & 0 \\ 0 & 0 & \bullet \end{bmatrix} \quad 2: \begin{bmatrix} \bullet & 0 & 0 & * \\ 0 & \bullet & 0 & * \\ 0 & 0 & \bullet & * \end{bmatrix} \quad 3: \begin{bmatrix} \bullet & 0 & * \\ 0 & \bullet & * \\ 0 & 0 & 0 \end{bmatrix} \quad 4: \begin{bmatrix} \bullet & 0 \\ 0 & \bullet \\ 0 & 0 \end{bmatrix}$$

1. The columns of A span $\mathbb{R}^3$: 1 and 2
2. T is one-to-one: 1 and 4
3. T is onto: 1 and 2
4. There is no $\mathbf{x}$ such that $A\mathbf{x} = 0$: None ($\mathbf{x} = \mathbf{0}$ always works)
5. T is neither one-to-one nor onto: just 3
6. The columns of A are linearly independent: 1 and 4 (same as "one-to-one")
7. $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution. 2 and 3 (same as "not one-to-one")
8. Any two columns of A are linearly dependent: None
9. $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^3$: 1 and 2 (same as "onto", same as "columns span $\mathbb{R}^3$")
10. There exists a $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is consistent: all of them ($\mathbf{b} = 0$ always works)
11. One of the columns of A can be deleted and A will still span $\mathbb{R}^3$: Just 2
12. There exists some $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent: 3 and 4 (same as "not onto", or "columns do not span $\mathbb{R}^3$")
13. There exists $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions: 2 and 3 (equivalent to "not one-to-one" for linear funtions)
14. Every $\mathbf{w} \in \mathbb{R}^3$ is a linear combination of the columns of A: 1 and 2 (same as "onto" or "columns span $\mathbb{R}^3$")

Miscellany

Let $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$. Sketch the set of all $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$ and the set of all $\mathbf{x}$ such that $A\mathbf{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.
If we say $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, then $A\mathbf{x} = \mathbf{0}$ is equivalent to $y = x/2$.
$A\mathbf{x} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ is equivalent to $y = x/2 - 1$, so the solution sets are two parallel lines with the solution set to the homogeneous equation passing through $(0, 0)$.
If we want to write the solutions in parametric form, then we could say the solution sets are

$$\left\{ t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$$

and

$$\left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\},$$

respectively.

Draw a capital "R" in the first quadrant of $\mathbb{R}^2$. What happens to the image if you apply the transformations $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, or $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$?

The first reflects the image across the y-axis, the second reflects it across the line $y = x$ (so you have an "R" on its back with its feet pointed toward the y-axis), and the third reflects it across the x-axis (so you have an upside-down "R" in the fourth quadrant)