10.2-10.3: Fourier Series and the Heat Equation

Tuesday, November 29

Heat Equation

Given the equations

1. $u_t = \beta u_{xx}$ for $0 < x < L$ and $t > 0$,
2. $u(0,t) = u(L,t) = 0$ for $t > 0$, and
3. $u(x,0) = f(x)$ for $0 < x < L$,

the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L},$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}.$$

1. Verify that any function of the form $u(x,t) = e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L}$ satisfies the first two conditions. How does this function behave as $t \to \infty$?

2. Solve the heat flow problem with $\beta = 3$, $L = \pi$, and $f(x) = \sin x - 6 \sin 4x$.

Sawtooth Wave

Define the function f on the interval $[0, 2\pi]$ by $f(x) = x - \pi$. Sketch the 2π-periodic extension of f, and find its Fourier series.

Legendre Polynomials

Consider the Legendre polynomials $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, which are orthogonal on the interval $[-1, 1]$. Find the "best" degree-2 approximation to $f(x) = \cos\left(\frac{\pi x}{2}\right)$ on this interval. How does it compare to the Taylor series?