10.2-10.3: Fourier Series and the Heat Equation

Tuesday, November 29

Heat Equation

Given the equations

1. $u_t = \beta u_{xx}$ for $0 < x < L$ and $t > 0$,
2. $u(0,t) = u(L,t) = 0$ for $t > 0$, and
3. $u(x,0) = f(x)$ for $0 < x < L$,

the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L},$$

where

$$f(x) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{L}.$$

1. Verify that any function of the form $u(x,t) = e^{-\beta(n\pi/L)^2 t} \sin \frac{n\pi x}{L}$ satisfies the first two conditions. How does this function behave as $t \to \infty$?

   ANSWER: since n is an integer, $\sin \frac{n\pi x}{L} = \sin n\pi = 0$ when $x = L$ and $\sin \frac{n\pi x}{L} = \sin 0 = 0$ when $x = 0$. Thus the boundary condition is satisfied.

   As for the diffusion rate,

   $$\begin{aligned} u_t &= -\beta(n\pi/L)^2 u(x,t) \\ u_{xx} &= -(n\pi/L)^2 u(x,t) \\ u_t &= \beta u_{xx}. \end{aligned}$$

   The given function $u(x,t)$ is therefore a solution to the heat equation.

2. Solve the heat flow problem with $\beta = 3$, $L = \pi$, and $f(x) = \sin x - 6 \sin 4x$.

   ANSWER:
   $u(x,t) = e^{-3t} \sin x - 6e^{-48t} \sin 4x$.

Sawtooth Wave

Define the function f on the interval $[0, 2\pi]$ by $f(x) = x - \pi$. Sketch the 2π-periodic extension of f, and find its Fourier series.
ANSWER: Since $f(x)$ is odd, we only need to look at the terms of the form $\sin(nx)$.

$$
\begin{aligned}
c_n &= \langle f(x), \sin nx \rangle \\
&= \frac{1}{\pi} \int_0^{2\pi} (x - \pi) \sin nx \, dx \\
&= \frac{1}{\pi} \left[\frac{-1}{n} x \cos nx + \frac{1}{n^2} \sin nx + \frac{\pi}{n} \cos nx \right]_0^{2\pi} \\
&= \frac{-2}{n}.
\end{aligned}
$$

Therefore, the Fourier series of $f(x)$ is $\sum_{n=1}^{\infty} \frac{-2}{n} \sin nx$.

Legendre Polynomials

Consider the Legendre polynomials $P_0(x) = 1, P_1(x) = x, P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$, which are orthogonal on the interval $[-1, 1]$. Find the "best" degree-2 approximation to $f(x) = \cos\left(\frac{\pi x}{2}\right)$ on this interval. How does it compare to the Taylor series?

First note that $\|P_0\|^2 = 2$ and $\|P_2\|^2 = \frac{5}{2}$, so we have to divide by these norms when computing c_0 and c_2.

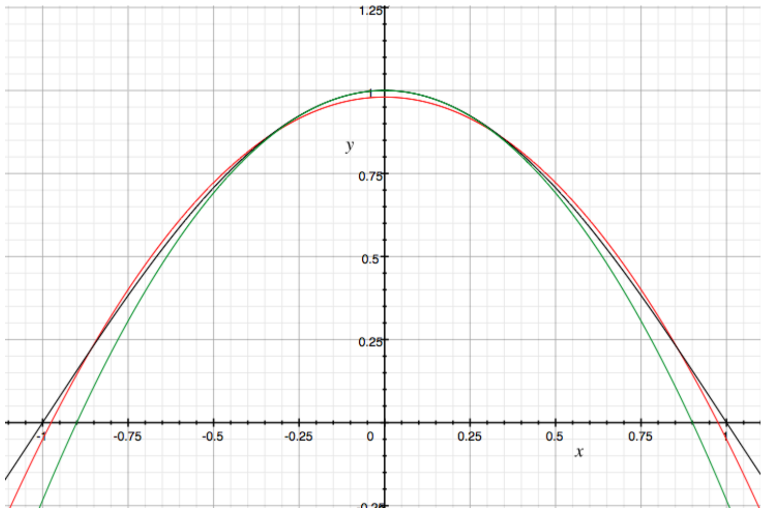
$$
\begin{aligned}
c_0 &= \frac{1}{2} \int_{-1}^{1} \cos \frac{\pi x}{2} \, dx \\
&= \frac{1}{\pi} [\sin \frac{\pi x}{2}]_{-1}^{1} \\
&= \frac{2}{\pi}.
\end{aligned}
$$

Since x is odd and $\cos x$ is even and the integral is symmetric about zero, $c_1 = 0$. Finally,

$$
\begin{aligned}
c_2 &= \frac{5}{2} \int_{-1}^{1} \left(\frac{3}{2}x^2 - \frac{1}{2} \right) \cos \frac{\pi x}{2} \, dx \\
&= \frac{10}{\pi} - \frac{120}{\pi^3}.
\end{aligned}
$$

The best degree-2 approximation to $f(x)$ on this interval is therefore $p(x) = (\frac{10}{\pi} - \frac{120}{\pi^3})(\frac{3}{2}x^2 - \frac{1}{2}) + \frac{2}{\pi}$. [Doing the computation isn't the most important part here; it's fine to use a computer to compute the integrals.]

A graph of both functions as well as the Taylor series for $f(x)$ is shown below. $f(x)$ is in black, $p(x)$ is in red, and the Taylor series is in green:



The Taylor series is a better approximation (in fact, the best approximation) at $x = 0$, but $p(x)$ is much better at the boundary of the interval. If we were to shrink the interval toward zero, then the best approximation $p(x)$ on the interval would converge to the Taylor series.