10.5-10.6: Heat Equation, Wave Equation

Thursday, December 1

Two-Dimensional Heat Equation

Consider the initial-boundary value problem

- $u_t = \beta \nabla^2 u$ for $0 < x < L, 0 < y < W$ and $t > 0$
- $u_x(0, y, t) = u_x(L, y, t) = 0$ for $0 < y < W$ and $t > 0$
- $u(x, 0, t) = u(x, W, t) = 0$ for $0 < x < L$ and $t > 0$
- $u(x, y, 0) = f(x, y)$ for $0 < x < L, 0 < y < W$.

Explain in words the physical problem being modeled by these conditions. If you assume a solution of the form $u(x, y, t) = X(x)Y(y)T(t)$, what conditions must X, Y, and T satisfy?

Wave Equation

- $u_{tt} = \alpha^2 u_{xx}$ for $0 < x < L$ and $t > 0$
- $u(0, t) = u(L, t) = 0$ for $t > 0$
- $u(x, 0) = f(x)$ for $0 < x < L$
- $u_t(x, 0) = g(x)$

Explain in words the implications of the four conditions, in particular the first. What properties of the string might affect α?

Verify that for any n, the function $u(x,t) = \left[a_n \cos \frac{n\pi\alpha}{L} t + b_n \sin \frac{n\pi\alpha}{L} t\right] \sin \frac{n\pi x}{L}$ satisfies the first two conditions. How does it relate to $f(x)$ and $g(x)$?

Plucked String

A string is lifted to a height h_0 at $x = a$ and released, giving initial conditions

$$f(x) = \begin{cases} h_0 x/a & 0 < x \leq a \\ h_0(L-x)/(L-a) & a < x < L \end{cases}$$

and $g(x) = 0$. Find a formal solution. Try finding the explicit solution where $a = L/2$.