10.5-10.6: Heat Equation, Wave Equation

Thursday, December 1

Two-Dimensional Heat Equation

Consider the initial-boundary value problem

- $u_t = \beta \nabla^2 u$ for $0 < x < L, 0 < y < W$ and $t > 0$
- $u_x(0, y, t) = u_x(L, y, t) = 0$ for $0 < y < W$ and $t > 0$
- $u(x, 0, t) = u(x, W, t) = 0$ for $0 < x < L$ and $t > 0$
- $u(x, y, 0) = f(x, y)$ for $0 < x < L, 0 < y < W$.

Explain in words the physical problem being modeled by these conditions. If you assume a solution of the form $u(x, y, t) = X(x)Y(y)T(t)$, what conditions must X, Y, and T satisfy?
ANSWER: The first condition describes the heat flow process, where the rate of change of the temperature at a point (x,y) is proportional to the difference between the temperature at that point and the average temperature of its neighbors. The second condition says that the walls in one direction are insulated, and the third says that the temperature of the other two walls is fixed. The fourth condition gives the initial temperature across the plate.
One condition, like for the 1-D heat problem, is that $T(t) = ce^{-\lambda t}$. So solutions of this form will stay proportional to their initial values, but will decrease exponentially over time. For more detail, this is example 4 in section 10.5.

Wave Equation

- $u_{tt} = \alpha^2 u_{xx}$ for $0 < x < L$ and $t > 0$
- $u(0, t) = u(L, t) = 0$ for $t > 0$
- $u(x, 0) = f(x)$ for $0 < x < L$
- $u_t(x, 0) = g(x)$

Explain in words the implications of the four conditions, in particular the first. What properties of the string might affect α?
ANSWER: The second equation says that the endpoints of the string are fixed and the third and fourth give initial conditions for the displacement and velocity. The first says that the acceleration (or force) at a point on the string will be proportional to its average displacement from its neighbors. Thus the restoring force will be strongest when the string is most bent and weakest when the string is close to straight.

Verify that for any n, the function $u(x,t) = \left[a_n \cos \frac{n\pi\alpha}{L} t + b_n \sin \frac{n\pi\alpha}{L} t\right] \sin \frac{n\pi x}{L}$ satisfies the first two conditions. How does it relate to $f(x)$ and $g(x)$?
ANSWER: The boundary conditons are met since $\sin n\pi x L = 0$ when $x = 0$ or $x = L$ (as n is an integer). As for the first equation, we can verify the following:

$$\begin{aligned} u_{tt} &= -(n\pi\alpha/L)^2 u(x,t) \\ u_{xx} &= -(n\pi/L)^2 u(x,t). \end{aligned}$$

Putting the two equations together implies that $u_{tt} = \alpha^2 u_{xx}$.

Plucked String

A string is lifted to a height h_0 at $x = a$ and released, giving initial conditions

$$f(x) = \begin{cases} h_0 x/a & 0 < x \leq a \\ h_0(L-x)/(L-a) & a < x < L \end{cases}$$

and $g(x) = 0$. Find a formal solution. Try finding the explicit solution where $a = L/2$.
ANSWER: The solution will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} b_n \cos \frac{n\pi\alpha}{L} \sin \frac{n\pi x}{L},$$

where

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

The coefficients b_n can then be found by using the Fourier transform. By symmetry, the integral is zero when n is even, so we assume that $n = 2k - 1$ for some integer k:

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L}\, dx \\ &= \frac{4}{L} \int_0^{L/2} h_0 x \frac{2}{L} \sin \frac{n\pi x}{L}\, dx \\ &= \frac{8}{L^2} h_0 \int_0^{L/2} x \sin \frac{n\pi x}{L}\, dx \\ &= \frac{8h_0}{L^2} \left[\frac{-L^2}{2\pi n} \cos \frac{n\pi}{2} + \left(\frac{L}{n\pi}\right)^2 \sin \frac{n\pi}{2} \right]_0^{L/2} \\ &= \frac{8h_0(-1)^k}{n^2\pi^2}. \end{aligned}$$