9.7-9.8: Nonhomogeneous systems, Matrix exponentials

Tuesday, November 22

Variation of Parameters

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t)\int \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds$$

Use the method of variaton of parameters given above to find a particular solution of the system

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}.$$

Also solve the problem using the method of undetermined coefficients. Which is simpler?
ANSWER: The matrix A has eigenvalues ± 1 with eigenvectors $v_1 = (-1, 3)$ and $v_2 = (-1, 1)$, so a fundamental matrix is $\mathbf{X}(t) = \begin{bmatrix} -e^{-t} & -e^t \\ 3e^{-t} & e^t \end{bmatrix}$, with

$$\mathbf{X}^{-1}(t) = \frac{1}{2}\begin{bmatrix} e^t & e^t \\ -3e^{-t} & -e^{-t} \end{bmatrix}.$$

Therefore, a particular solution is given by

$$\begin{aligned}
\mathbf{X}(t)\int \mathbf{X}^{-1}(s)\mathbf{f}(s)\,ds &= \mathbf{X}(t)\int \frac{1}{2}\begin{bmatrix} e^s & e^s \\ -3e^{-s} & -e^{-s} \end{bmatrix}\begin{bmatrix} 2e^s \\ 4e^s \end{bmatrix} ds \\
&= \mathbf{X}(t)\int \begin{bmatrix} 3e^{2s} \\ -5 \end{bmatrix} ds \\
&= \begin{bmatrix} -e^{-t} & -e^t \\ 3e^{-t} & e^t \end{bmatrix}\begin{bmatrix} \frac{3}{2}e^{2t} \\ -5t \end{bmatrix} \\
&= \begin{bmatrix} 5te^t - \frac{3}{2}e^t \\ \frac{9}{2}e^t - 5te^t \end{bmatrix}.
\end{aligned}$$

ANSWER 2: Notice that the fundamental matrix contains a solution of the form $e^t\mathbf{a}(t)$, so for undetermined coefficients we have to go one step further, and guess

$$\mathbf{x}(t) = \begin{bmatrix} ate^t + be^t \\ cte^t + de^t \end{bmatrix},$$

in which case we get the system

$$\begin{bmatrix} ate^t + (a+b)e^t \\ cte^t + (c+d)e^t \end{bmatrix} = \begin{bmatrix} (2a+c)te^t + (2b+d)e^t \\ (-3a-2c)te^t + (-3b-d)e^t \end{bmatrix} + \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ -3 & 0 & -3 & 0 \\ 0 & -3 & -1 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -1 & -3 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -10 \end{bmatrix},$$

which gives the set of solutions $a = 5, c = -5, b + d = 3$. Any such choice of b and d will do, and a look at the solution given by variation of parameters can note that $b = -3/2, d = 9/2$ does work.

Use the method of undetermined coefficients to find a particular solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix}.$$

ANSWER: guess a solution of the form

$$\mathbf{x}(t) = \begin{bmatrix} a\cos t + b\sin t \\ c\cos t + d\sin t \end{bmatrix},$$

in which case we get the problem

$$\begin{bmatrix} b\cos t - a\sin t \\ d\cos t - c\sin t \end{bmatrix} = \begin{bmatrix} c\cos t + d\sin t \\ (3c - 2a)\cos t + (3d - 2b)\sin t \end{bmatrix} + \begin{bmatrix} \sin t \\ \cos t + \sin t \end{bmatrix},$$

which can be turned into the linear system of equations

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & -1 \\ 2 & 0 & -3 & 1 \\ 0 & 2 & -1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

which gives the solution $(a, b, c, d) = (-\frac{2}{5}, -\frac{4}{5}, -\frac{4}{5}, -\frac{3}{5})$, or $\mathbf{x}(t) = \begin{bmatrix} -\frac{2}{5}\cos t - \frac{4}{5}\sin t \\ -\frac{4}{5}\cos t - \frac{3}{5}\sin t \end{bmatrix}$.

The Matrix Exponential: Recap

- e^{At} is a fundamental matrix for the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$.
- If $X(t)$ is a fundamental matrix then $e^{At} = X(t)X(0)^{-1}$.
- If $A = V\Lambda V^{-1}$ then $e^{A} = Ve^{\Lambda}V^{-1}$.

Find e^{At}, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$.
ANSWER: We can find the eigenvalue/eigenvector factorization

$$A = V\Lambda V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

so

$$e^{A} = Ve^{\Lambda}V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e+e^3 & e-e^3 \\ e-e^3 & e+e^3 \end{bmatrix}.$$

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, what are the eigenvectors of A and B? Find formulas for A^n and B^n. What are e^A and e^B?
ANSWER: A and B both have 1 as their only eigenvalue and $\mathbf{e}_1$ as their only eigenvector. $A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ and $B^n = \begin{bmatrix} 1 & n & (n^2-n)/2 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}$.
To find e^A, split A as $A = I + N$ where $N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (so that $N^2 = 0$). Then $e^A = e^{I+N} = e^I e^N = e \cdot (I + N) = \begin{bmatrix} e & e \\ 0 & e \end{bmatrix}$. Similarly, $B = \begin{bmatrix} e & e & e/2 \\ 0 & e & e \\ 0 & 0 & e \end{bmatrix}$.