9.5: Homogeneous Linear Systems

Tuesday, November 15

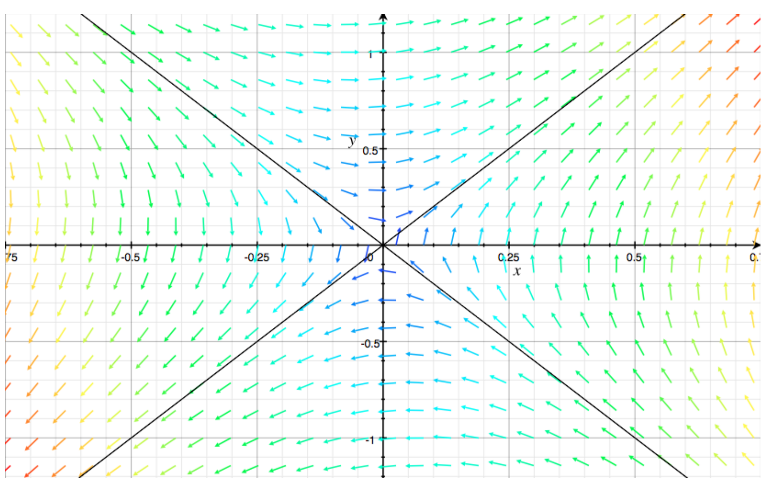
Recap

If an $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$, then $\{e^{\lambda_1 t}\mathbf{u}_1, \ldots, e^{\lambda_n t}\mathbf{u}_n\}$ is a fundamental solution set for the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

Practice Problems

1. If $A = \begin{bmatrix} 1 & 3 \\ 12 & 1 \end{bmatrix}$, find the general solution to $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
2. If we make the interpretation $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, sketch a direction field for the differential equation.
3. If a particle starts at the point $(-1, 2)$, sketch the trajectory of the solution. What happens to the particle as $t \to \infty$?
4. If a particle starts at $(-1, 1,)$, sketch the trajectory of the solution. What happens to the particle as $t \to \infty$?
5. If you have time/space, do the same with $\mathbf{A} = \begin{bmatrix} -1 & \frac{3}{4} \\ -5 & 3 \end{bmatrix}$.

For the first matrix given, the eigenvalues are $\lambda = -5, 7$ with eigenvectors $(1, -2)$ and $(1, 2)$, repsectively. If the particle starts on the line $y = -2x$, it will therefore approach the origin as $t \to \infty$. If it begins everywhere else, it will move away from the origin and gradually approach the line $y = 2x$, the path corresponding to the eigenvector with eigenvalue 7.



Simple Harmonic Oscillator

Say we have an object with mass $m = 1$ attached to a spring with stiffness $k = 1$ in a frictionless system. Then we can derive the system of equations

$$\begin{bmatrix} x(t) \\ v(t) \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}.$$

1. Find the eigenvalues and eigenvectors of A.
2. Sketch a direction field for the system of differential equations.
3. Find the general solution to the system.
4. Find the specific solution given the intitial conditions $x(0) = 2, v(0) = 0$.
5. If the object has kinetic energy $\frac{1}{2}mv^2$ and potential energy $\frac{1}{2}kx^2$, show that the total energy of the system is constant. Does this hold in a system with friction? How can you interpret this result geometrically with respect to the direction field?

The specific solution is $x(t) = 2\cos t$, $v(t) = -2\sin t$. The direction field consists of concentric circles going clockwise around the origin. Since $\frac{1}{2}mv^2 + \frac{1}{2}kx^2$ is a constant for any starting $x(0)$ and $v(0)$, this system always conserves energy. If there is friction, it does not.