9.6-9.7: Complex Eigenvalues, Variation of Parameters

Thursday, November 17

Recap

If a real matrix A has complex eigenvectors $\mathbf{x} \pm i\mathbf{y}$ with complex eigenvalues $\alpha \pm i\beta$, then two real solutions to the system $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{x}_1(t) = e^{\alpha t} \cos \beta t \mathbf{x} - e^{\alpha t} \sin \beta t \mathbf{y}$ and $\mathbf{x}_2(t) = e^{\alpha t} \sin \beta t \mathbf{x} + e^{\alpha t} \cos \beta t \mathbf{y}$.

Coupled Mass-Spring System

Say we have the coupled mass-spring system governed by the equations

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1), \\ m_2 x_2'' &= -k_2(x_2 - x_1) - k_3 x_2 \end{aligned}$$

with $m_1 = m_2 = 1$ kg, $k_1 = k_2 = 2$ N/m, and $k_3 = 3$ N/m. Determine the normal frequencies for this coupled mass-spring system.

Variation of Parameters

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t) \int \mathbf{X}^{-1}(s)\mathbf{f}(s)\, ds$$

Use the method of variaton of parameters given above to find a general solution of the system

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}.$$

Complex Eigenvalues

Suppose that the *real* matrix A has a complex eigenvalue $\mathbf{v} = \mathbf{x} + i\mathbf{y}$ with complex eigenvector $\lambda = \alpha + i\beta$.

1. Compare real and imaginary parts to show that $Ax = \alpha x - \beta y$ and $Ay = \beta x + \alpha y$.
2. Show that $A \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{x} & \mathbf{y} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$.
3. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$.
4. With respect to the basis $\mathcal{B} = \{e^{it}, e^{-it}\}$, what is $[D]_{\mathcal{B}}$?
5. What is $[D]_{\mathcal{C}}$ with respect to the basis $\mathcal{C} = \{\sin t, \cos t\}$?
6. What is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$?
7. Diagonalize the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.