9.6-9.7: Complex Eigenvalues, Variation of Parameters

Thursday, November 17

Recap

If a real matrix A has complex eigenvectors $\mathbf{x} \pm i\mathbf{y}$ with complex eigenvalues $\alpha \pm i\beta$, then two real solutions to the system $\mathbf{x}' = A\mathbf{x}$ are $\mathbf{x}_1(t) = e^{\alpha t} \cos \beta t \mathbf{x} - e^{\alpha t} \sin \beta t \mathbf{y}$ and $\mathbf{x}_2(t) = e^{\alpha t} \sin \beta t \mathbf{x} + e^{\alpha t} \cos \beta t \mathbf{y}$.

Coupled Mass-Spring System

Say we have the coupled mass-spring system governed by the equations

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1), \\ m_2 x_2'' &= -k_2(x_2 - x_1) - k_3 x_2 \end{aligned}$$

with $m_1 = m_2 = 1$ kg, $k_1 = k_2 = 2$ N/m, and $k_3 = 3$ N/m. Determine the normal frequencies for this coupled mass-spring system.
ANSWER: if we set $y_1 = x_1, y_2 = x_1', y_3 = x_2, y_4 = x_2'$, then we get the system

$$\mathbf{y}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -5 & 0 \end{bmatrix} \mathbf{y}(t)$$

This system has eigenvalues $\pm \frac{i}{2}\sqrt{9 \pm \sqrt{17}}$, so the two normal frequencies are $\frac{\sqrt{9 \pm \sqrt{17}}}{4\pi}$ cycles per second.

Variation of Parameters

$$\mathbf{x}(t) = \mathbf{X}(t)\mathbf{c} + \mathbf{X}(t) \int \mathbf{X}^{-1}(s)\mathbf{f}(s)\, ds$$

Use the method of variaton of parameters given above to find a general solution of the system

$$\mathbf{x}'(t) = \begin{bmatrix} 2 & 1 \\ -3 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix}.$$

ANSWER: The matrix A has eigenvalues ± 1 with eigenvectors $v_1 = (1, -3)$ and $v_2 = (1, -1)$, so a fundamental matrix is $\mathbf{X}(t) = \begin{bmatrix} e^{-t} & e^t \\ -3e^{-t} & -e^t \end{bmatrix}$, with

$$\mathbf{X}^{-1}(t) = \frac{1}{2} \begin{bmatrix} -e^t & -e^t \\ 3e^{-t} & e^{-t} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathbf{X}(t) \int \mathbf{X}^{-1}(s)\mathbf{f}(s)\, ds &= \mathbf{X}(t) \int \frac{1}{2} \begin{bmatrix} -e^t & -e^t \\ 3e^{-t} & e^{-t} \end{bmatrix} \begin{bmatrix} 2e^t \\ 4e^t \end{bmatrix} ds \\ &= \mathbf{X}(t) \int \begin{bmatrix} -3e^{2t} \\ 5 \end{bmatrix} \\ &= \mathbf{X}(t) \begin{bmatrix} -\frac{3}{2}e^{2t} \\ 5t \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2}e^t + 5te^t \\ \frac{9}{2}e^t + 5te^t \end{bmatrix}. \end{aligned}$$

Well, that was painful. Maybe try using undetermined coefficients next time instead.

Complex Eigenvalues

Suppose that the *real* matrix A has a complex eigenvalue $\mathbf{v} = \mathbf{x} + i\mathbf{y}$ with complex eigenvector $\lambda = \alpha + i\beta$.

1. Compare real and imaginary parts to show that $Ax = \alpha x - \beta y$ and $Ay = \beta x + \alpha y$.

   ANSWER: $A(\mathbf{x} + i\mathbf{y}) = (\alpha + i\beta)(\mathbf{x} + i\mathbf{y}) = (\alpha x - \beta y) + i(\beta x + \alpha y)$, so the real and imaginary parts of both sides must be equal.

2. Show that $A\begin{bmatrix}\mathbf{x} & \mathbf{y}\end{bmatrix} = \begin{bmatrix}\mathbf{x} & \mathbf{y}\end{bmatrix}\begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix}$.

   ANSWER: Follows immediately from the previous answer by stacking the vectors.

3. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix}$.

   ANSWER: The matrix has eigenvalues $\alpha \pm i\beta$, with eigenvectors $(1, i)^T$ and $(1, -i)^T$, respectively.

4. With respect to the basis $\mathcal{B} = \{e^{it}, e^{-it}\}$, what is $[D]_{\mathcal{B}}$?

   ANSWER: $[D]_{\mathcal{B}} = \begin{bmatrix}i & 0 \\ 0 & -i\end{bmatrix}$.

5. What is $[D]_{\mathcal{C}}$ with respect to the basis $\mathcal{C} = \{\sin t, \cos t\}$?

   ANSWER: $[D]_{\mathcal{C}} = \begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$

6. What is the change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$?

   ANSWER:

$$\begin{bmatrix}e^{it} \\ e^{-it}\end{bmatrix} = \begin{bmatrix}\cos t + i\sin t \\ \cos t - i\sin t\end{bmatrix} = \begin{bmatrix}1 & i \\ 1 & -i\end{bmatrix}\begin{bmatrix}\cos t \\ \sin t\end{bmatrix},$$

   so the change of basis matrix is $P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix}$.

7. Diagonalize the matrix $\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}$.

   ANSWER:

$$\begin{bmatrix}0 & -1 \\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix} = \begin{bmatrix}1 & 1 \\ i & -i\end{bmatrix}\begin{bmatrix}-i & 0 \\ 0 & i\end{bmatrix}.$$