4.6: Variation of Parameters

Tuesday, November 8

Recap

Suppose we want to solve $ay'' + by' + cy = f$:

1. Find linearly independent solutions y_1, y_2 to the homogeneous equation.
2. Look for a solution of the form $y = v_1 y_1 + v_2 y_2$.
3. Add the constraint $v_1' y_1 + v_2' y_2 = 0$. This gives the system

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f/a \end{bmatrix}.$$

4. Since y_1 and y_2 are LI, the Wronskian $W(t) = (y_1 y_2' - y_1' y_2)$ is always nonzero. Solve the system and integrate to get

$$v_1(t) = \int \frac{-f(t)y_2(t)}{aW(t)} + c_1$$
$$v_2(t) = \int \frac{f(t)y_1(t)}{aW(t)} + c_2.$$

5. Note that this will be a solution for any constants c_1 and c_2.

Practice Problems

Solve the following problems first by using undetermined coefficients, and then by variation of parameters. Which method was quicker?

1. $y'' - y = 2t + 4$:

   ANSWER: guess $y = At + B$, get $y = -2t - 4$. Very easy!

   Variation of parameters involves $y_1 = e^t, y_2 = e^{-t}$.

2. $y'' - 2y' - 4y = 2e^{2t}$

   ANSWER: Guess an equation of the form $y = Ae^{2t}$ and get $4A - 4A - 4A = 2$, or $A = -\frac{1}{2}$. Not too bad!

   Variation of parameters is a good bit harder.

Find a general solution to the differential equation $y'' + y = \sec t$ using variation of parameters.
ANSWER: if we use the independent solutions $y_1 = \sin t, y_2 = \cos t$, then the Wronskian is a constant -1.

Linear Independence of Functions

Show that the functions $p_1 = t, p_2 = t^2$, and $p_3 = 1 - t^2$ are linearly independent by finding a set of three points $\{x_1, x_2, x_3\} \subset \mathbb{R}$ on which they are linearly independent.
ANSWER: the set $\{-1, 0, 1\}$ will do.

Linear Operators

Say we want to find a particular solution to the differential equation $y'' - 2y' - 3y = t^2 + t + 2$. If D is the derivative operator and we restrict our attention to the vector space $V = \text{Span}\mathcal{B} = \text{Span}\{1, t, t^2\}$, find $[D]_{\mathcal{B}}$.

ANSWER: $[D]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Rewrite the equation in the form $Ly = p$, where L is a single linear operator expressed in terms of D and I, the identity operator. Write it again with respect to the basis $\mathcal{B}$.

ANSWER: $y'' - 2y' - 3y = (D^2 - 2D - 3)y$. $[L]_{\mathcal{B}} = \begin{bmatrix} -3 & -2 & 2 \\ 0 & -3 & -4 \\ 0 & 0 & -3 \end{bmatrix}$, which is invertible. Therefore there is a unique solution in our chosen vector space. (This does not mean that the differential equation has a unique solution, because the solutions to the homogeneous equation are of the form $y = c_1 e^t + c_2 e^{3t}$.)