6, 9.4: Higher-Order Equations

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Linear Operators and Linear Independence

Find a differential equation for which $\{1, \sin t, \cos t\}$ is a fundamental solution set. Use the Wronskian to demonstrate that these three functions are linearly independent.
ANSWER: All are solutions to $D(D^2 + 1)y = 0$, or $y''' + y' = 0$. As for the Wronskian,

$$\begin{vmatrix} 1 & \sin t & \cos t \\ 0 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} = \begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix} = -1.$$

Since this is nonzero at all points, the functions are linearly independent.

Show that $(D-1)(D+2)[t^2 - 2t + 5] = (D+2)(D-1)[t^2 - 2t + 5]$.
ANSWER: The short idea is that both of these are equal to the operator $D^2 - D - 2$, which really just uses the fact that if I is the idenity operator then $ID = DI$. If we expand these out in full, then we get

$$(D-1)(D+2)[t^2 - 2t + 5] = (D-1)(2t^2 - 2t + 8) = -2t^2 - 6t - 10$$

and

$$(D+2)(D-1)[t^2 - 2t + 5] = (D+2)(-t^2 + 4t - 7) = -2t^2 - 6t - 10,$$

so the two are equal.

If D is the derivative operator, find a general solution to the differential equation

$$(D-1)^2(D+3)(D^2 + 2D + 5)^2[y] = 0.$$

ANSWER: $y = c_1 e^t + c_2 t e^t + c_3 e^{-3t} + c_4 e^{-t} \sin 2t + c_5 e^{-t} \cos 2t$.

Show that the functions e^t, t^2, and $\sin t$ are linearly independent by finding for each term a linear operator that annihilates the other two terms but not that one. Why does this show that the functions are linearly independent?
ANSWER: Each one is individually annihilated by $D-1, D^3$, and D^2+1, so each of the three combinations $(D-1)D^3, (D-1)(D^2+1), D^3(D^2-1)$ annihilate two but not the third.
This shows that they are linearly independent because the null space of any linear operator is a subspace. Thus if two vectors are in a null space but a third is not, the third vector is linearly independent from the other two.

Matrix Methods for Linear Systems

Write the given system in the matrix form $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$:

$$x'(t) = 3x(t) - y(t) + t^2$$
$$y'(t) = -x(t) + 2y(t) + e^t$$

ANSWER: if $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, then we can write this system as the equation

$$\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

Write the system $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \mathbf{x} + e^t \begin{bmatrix} t \\ 1 \end{bmatrix}$ as a set of scalar equations.
ANSWER:

$$x_1(t) = 2x_1(t) + x_2(t) + te^t$$
$$x_2(t) = -x_1(t) + 3x_2(t) + e^t$$

Verify that the vector functions $\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$ are solutions to the system $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$.
ANSWER:

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \mathbf{x}_1'.$$

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -3e^{-t} \end{bmatrix} = \mathbf{x}_2'.$$