4.1-4.2: Linear Second-Order Equations

Tuesday, November 1

Second-order Linear Equations

Verify that the function $y(t) = e^{-3t}\sin(\sqrt{3}t)$ is a solution to the equation $my'' + by' + ky = 0$ if $m = 1, b = 6, k = 12$. What happens to the solution as $t \to \infty$?
ANSWER: as $t \to \infty$, $y(t)$ continues oscillating around zero but the size of the oscillations shrinks to zero.

Find the solution to the initial value problem $y'' - 4y' + 3y = 0$, $y(0) = 1$, $y'(0) = 1/3$.
ANSWER: General solution is $y = c_1 e^t + c_2 e^{3t}$, solve from there.

Find the solution to the initial value problem $y'' - 4y' + 4y' = 0$, $y(1) = 1$, $y'(1) = 1$.
ANSWER: General solution is $y = c_1 e^{2t} + c_2 t e^{2t}$, solve from there.

Linear Independence and Subspaces

If $C^\infty(\mathbb{R})$ is the vector space of infinitely-differentiable functions on $\mathbb{R}$, show that for fixed a, b, c the set of solutions to the differential equation $ay'' + by' + cy = 0$ is a subspace of $C^\infty(\mathbb{R})$. Better yet: show that the solution set is the null space of a particular linear transformation.
ANSWER: we could check the subspace axioms directly, but easier would be to use the hint. D the derivative operator, is a linear transformation from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$, and so is $(aD^2 + bD + cI)$. The solution set is the null space of this linear transformation, and is therefore a subspace.

If f and g are linearly dependent functions, show that $(f/g)' = 0$ whenever $g \neq 0$. How does this relate to the Wronskian of f and g?
ANSWER: If $f = cg$, then $f/g = c$ so $(f/g)' = 0$, defined whenever $g \neq 0$. We can expand this as $(f/g)' = (gf' - fg')/g^2$, so whenever $g \neq 0$ the numerator (the Wronskian) must also be zero.

Show that the three functions $y_1(t), y_2(t) = te^t, y_3(t) = t^2 e^t$ are linearly independent on $\mathbb{R}$.
ANSWER: easiest would probably be to show that they are linearly independent on the set $(-1, 0, 1)$ or something like that: evaluated at these three points, we get the vectors $(1/e, 1, e), (-1/e, 0, e), (1/e, 0, e)$. The second and third are linearly independent, and the first is independent from the other two because of the second coordinate.

Bonus: Show that any two solutions to the differential equation $ay' + by = 0$ must be linearly dependent.
ANSWER: Show that $y = e^{-(b/a)t}$ is a solution and that for any other solution f, the Wronskian of f and y is zero.