4.3-4.5: Linear Second-Order Equations

Thursday, October 27

4.3: Complex Roots

Warmup: what is $e^{i\pi/3}$ in terms of sines and cosines?
ANSWER: $e^{i\pi/3} = \cos \pi/3 + i \sin \pi/3 = \frac{1}{2} + i\sqrt{3}/2$.

Solve the initial value problem $w'' - 4w' + 2w = 0$ where $w(0) = 0, w'(0) = 1$.
ANSWER: The characteristic equation has roots $2 \pm i\sqrt{3}$, so the solution is of the form $w = c_1 e^{2t} \cos \sqrt{3}t + c_2 e^{2t} \sin \sqrt{3}t$. If $w(0) = 0$ then $c_1 = 0$, and if $w'(0) = 1$ then $c_2 = 1/\sqrt{3}$.

4.4: Nonhomogeneous Equations

Explain what the solution will look like for a differential equation of the form...

- $ay'' + by' + cy = p_k(t)$, where p_k is a degree k polynomial.

  ANSWER: a degree k polynomial.

- $ay'' + by' + cy = p_k(t)e^{rt}$

  ANSWER: $y = q_k(t)e^{rt}$, possibly up to $y = q_{k+2}e^{rt}$ (with a degree $k + 2$ polynomial) depending on whether r is a root of the characteristic equation.

- $ay'' + by' + cy = e^{\alpha t} \cos \beta t$

  $y = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$.

Find particular solutions to the given differential equations:

1. $y'' + 2y' - y = 10$

   ANSWER: Make a guess that y is a degree-0 polynomial, so $y = C$. Solving gives $y = -10$.

2. $y'' + 4y = 16t \sin 2t$

   ANSWER: Guess $y = A_2 t^2 \sin 2t + A_1 t \sin 2t + B_2 t^2 \cos 2t + B_1 t \sin 2t$, since $\sin 2t$ and $\cos 2t$ are roots of the homogeneous equation. Have fun solving it!

4.5 In One Sentence

The set of solutions to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x}_* + \text{Nul}(A)$, where $\mathbf{x}_*$ is any particular solution.

Find all solutions of the equation $y'' + y = 1$.
ANSWER: $y = 1 + c_1 \cos t + c_2 \sin t$.

The Derivative Operator

Say we want to solve the differential equation $y'' + 2y' + 4y = 5\sin 3t$, and suspect that the solution will be of the form $A\sin 3t + B\cos 3t$. If $\mathcal{B} = \{\sin 3t, \cos 3t\}$ and D is the derivative operator, find $[D]_{\mathcal{B}}$ and $[D^2]_{\mathcal{B}}$. Use this information to set up the problem as a system of equations $A\mathbf{x} = \mathbf{b}$.
ANSWER: $D = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$, $D^2 = -9I$.
Then $y'' + 2y' + 4y = (D^2 + 2D + 4) = 2D - 5 = \begin{bmatrix} -5 & -6 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$.

Say we want to solve the differential equation $y'' - 2y' + y = 3e^t$. If $\mathcal{B}$ is the basis $\{e^t\}$, what does the resulting linear system look like? If $\mathcal{C}$ is the basis $\{te^t, e^t\}$, what are $[D]_{\mathcal{C}}$ and the resulting linear system?
ANSWER: $[D]_{\mathcal{B}}$ and $[D]_{\mathcal{C}}$ are both zero matrices. We need to go one step higher and look for a solution in the span of $\{t^2e^t, te^t, e^t\}$.

Linear Independence

Prove that if T is a linear transformation and $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent set, then $\{v_1, \ldots, v_n\}$ is also a linearly independent set.
ANSWER: If $c_1T(v_1) + \ldots + c_nT(v_n) = 0$ implies that $c_1, \ldots c_n = 0$, then

$$\begin{aligned} d_1v_1 + \ldots + d_nv_n = 0 &\Rightarrow T(d_1v_1 + \ldots + d_nv_n) = 0 \\ &\Rightarrow d_1T(v_1) + \ldots + d_nT(v_n) = 0 \\ &\Rightarrow d_1, \ldots d_n = 0. \end{aligned}$$

If we put everything in matrix form, then this says that if AV has full column rank then so does V. (The contrapositive may make more sense: if V has a nontrivial null space then so does AV.)

For $p \in \mathbb{P}_2$, define $T : \mathbb{P}_2 \to \mathbb{R}^3$ by $T(p) = [p(0), p(1), p(3)]$. Use the previous problem to show that $\{1, t, t^2\}$ is a linearly independent set.
ANSWER:
$T(1) = [1, 1, 1], T(2) = [0, 1, 3], T(3) = [0, 3, 9]$. These vectors in $\mathbb{R}^3$ are linearly independent, and therefore so are the original vectors.

Show that $\{\sin t, \sin 2t, \cos t\}$ is a linearly independent set.
ANSWER: Evaluating at $0, \pi/4, \pi/2$ works.