1.1-1.2: Row Reduction and Echelon Forms: Solutions

Thursday, August 25

Sets

Describe each of the following sets:

- $\mathbb{N}$: Natural numbers: $\{1, 2, 3, \ldots\}$
- $\mathbb{Z}$: Integers: $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- $\mathbb{R}$: All real numbers
- $\mathbb{C}$: Complex numbers: $\{x + iy : x, y \in \mathbb{R}\}$, where $i^2 = -1$
- $\mathbb{Z} \cap \mathbb{R}$: Just $\mathbb{Z}$, since $\mathbb{Z} \subset \mathbb{R}$
- $\mathbb{Q} \cup \mathbb{N}$: Just $\mathbb{Q}$ (all rational numbers), since $\mathbb{N} \subset \mathbb{Q}$
- $\mathbb{R}^2$: All ordered pairs $\{(x, y) : x, y \in \mathbb{R}\}$. Also describes 2-dimensional space.
- $\{n + \frac{1}{2} : n \in \mathbb{N}\}$: $\{1\frac{1}{2}, 2\frac{1}{2}, \ldots\}$
- $\{x : x \in \mathbb{R}, x^2 < 1\}$: $(-1, 1)$.

Use set builder notation to describe the set of all odd integers.

$$\{2z + 1 : z \in \mathbb{Z}\}$$

Sketch the following subset of $\mathbb{R}^2$: $[(\frac{t}{2}, t + 1) : t \in \mathbb{R}]$
Equivalent to the line described by $y = 2x + 1$ for all $x \in \mathbb{R}$.

Row operations

What are the 3 types of elementary row operations on a matrix?

- Muliply a row by any non-zero real number
- Switch two rows
- Add a multiple of any one row to any other row

All of these operations are reversible.

Give an example of a matrix that is in echelon form but not reduced echelon form. Give an example of a matrix whose entries are all 1 or 0 but is not in echelon form.
For the first example, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not in reduced form since the second pivot element is not the only non-zero in its column.

For the second example, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not in echelon form since there is a pivot element that appears above a non-zero element.

(1.2, Example 3): Use elementary row operations to transform the following matrix into echelon form, then reduced echelon form.

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \mapsto_{(1)\leftrightarrow(3)} \begin{bmatrix} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix}$$

$$\mapsto_{(1)=(1)/3} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \end{bmatrix}$$

$$\mapsto_{(3)=(3)-3\cdot(1)} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 3 & -6 & 6 & 4 & -5 \\ 0 & 2 & -4 & 4 & 2 & -6 \end{bmatrix}$$

$$\mapsto_{(3)\leftrightarrow(2)} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\mapsto_{(2)=(2)/2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

$$\mapsto_{(3)=(3)-3\cdot(1)} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\mapsto_{(2)=(2)-(3)} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\mapsto_{(1)=(1)-2\cdot(3)} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\mapsto_{(1)=(1)+3\cdot(2)} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

(1.2, Example 5): Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

Put it in reduced echelon form (this was already done above):

$$\begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Solutions exists, but are not unique. We get the following description of the solution set:

$$\begin{aligned} &x_1 = -24 + 2x_3 - 3x_4 \\ &x_2 = -7 + 2x_3 - 2x_4 \\ &x_3 \text{ is free} \\ &x_4 \text{ is free} \\ &x_5 = 4 \end{aligned}$$

We have two free variables.

Suppose you want to find a parabola of the form $y = ax^2 + bx + c$ that passes through the points (1,1), (-1,7), and (2,4). Set up this problem as a system of linear equations, form the augmented matrix system and transform it to reduced echelon form, and describe the set of solutions.

$$\begin{aligned} a(1)^2 + b(1) + c &= 1 \\ a(-1)^2 + b(-1) + c &= 7 \\ a(2)^2 + b(2) + c &= 4 \end{aligned}$$

This gives us the augmented matrix

$$\begin{aligned} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 7 \\ 4 & 2 & 1 & 4 \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & 6 \\ 0 & -2 & -3 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & -3 & -6 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{aligned}$$

The (unique) parabola is therefore given by $y = 2x^2 - 3x + 2$.

Do the same with fitting a parabola of the form $y = ax^2 + bx + c$ through the following sets of points:

- (-1,2), (2,3), and (2,5)

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 4 & 2 & 1 & 3 \\ 4 & 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 4 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

  We can stop here, since the last row shows that the system is inconsistent. Intuitively, there is no such parabola since the points (2,3) and (2,5) have the same x coordinate but different y coordinates, which no function can satisfy.

- (-2,8), (-1,4), (0,3), and (2,4)

$$\begin{bmatrix} 4 & -2 & 1 & 8 \\ 1 & -2 & 1 & 4 \\ 0 & 0 & 1 & 3 \\ 4 & 2 & 1 & 4 \end{bmatrix} \sim_{(3)\ldots} \begin{bmatrix} 4 & -2 & 0 & 5 \\ 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$
$$\sim_{(4)\ldots} \begin{bmatrix} 8 & 0 & 0 & 6 \\ 5 & 0 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

  In this case, the first two rows not contradict each other. (The row reduction may not have been fully explained here but no matter how you do it you will get some contradiction.) The basic lesson is that three points is "just right" for determining a parabola but four is "too many".

- (-1,2) and (1,2)

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 0 & 0 \end{bmatrix},$$

  so we get the constraints $b = 0$, $a + c = 2$. This means that there are infinitely many parabolas that go through these points, but all are of the form $ax^2 + c$, where $a + c = 2$. So two points is "not enough" to determine a parabola uniquely.