Midterm 2: Review

Tuesday, October 25

1 Definitions

- Null space, column space
- basis
- dimension, isomorphism
- Rank Theorem (Rank-Nullity Theorem)
- Row Space
- Change-of-basis matrix
- Eigenvector, eigenvalue, eigenspace
- characteristic polynomial
- similarity, diagonalizability
- dot product, orthogonality, norm
- projection, perpendicular space
- orthogonal sets, orthonormal sets
- Gram-Schmidt, QR, orthogonal matrix
- Least-squares problem, normal equations
- Use of QR in solving least squares
- Inner product
- symmetric matrix, Spectral Theorem

2 Computational Problems

- Given a vector $\mathbf{x}$ and basis $\mathcal{B}$, find $[\mathbf{x}]_{\mathcal{B}}$.
- Determine whether two given vector spaces are isomorphic.
- Given the size and column space of a matrix, find the dimensions of the row space and null space (or vice versa).
- Given two bases $\mathcal{B}$ and $\mathcal{C}$, find the change-of-basis matrices $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$.
- Find the eigenvectors and eigenvalues of a 2×2 or 3×3 matrix.
- Determine whether a given matrix is diagonalizable.
- Given a factorization $A = PDP^{-1}$, find A^4 and A^{-1} in terms of P and D.
- Given a matrix A and a basis $\mathcal{B}$, find $[A]_{\mathcal{B}}$.
- Find the distance between two given vectors.

- Find the perpendicular space for a given subspace.
- Given a vector $\mathbf{y}$ and a subspace W, find the distance from $\mathbf{y}$ to W and the projection of $\mathbf{y}$ onto W.
- Find an orthogonal basis for a given set of vectors.
- Given an inner product $\langle , \rangle$ for a vector space V, (1) compute the inner product of two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, (2) find an orthogonal basis for a set of vectors.

3 Conceptual Problems/Quick Checks

- What is a basis? What is the relation between bases of $\mathbb{R}^n$ and invertible matrices?
- How do the dimensions of the row space, column space, and null space of a matrix relate to each other?
- How can you quickly tell whether a matrix is rank zero? Rank 1?
- How can you quickly tell whether a 2×2 matrix is orthogonal?
- What is the formula for the inverse of a 2×2 matrix?
- What is the algebraic multiplicity of an eigenvalue of a matrix? How does it relate to the dimension of the eigenspace for that eigenvalue?
- If a matrix is not invertible, what can you conclude about its eigenvalues?
- If a matrix is not diagonalizable, what can you conclude about its eigenvalues?
- UNIQUE REPRESENTATION: how does this concept apply to (1) a basis for a vector space, (2) A subspace W and its perpendicular space $W^\perp$, (3) an orthogonal basis?
- BE ABLE TO SKETCH WHAT A PROJECTION LOOKS LIKE IN 2 OR 3 DIMENSIONS.
- What is the relation between the perpendicular space of a matrix and its column/row/null spaces? How do the dimensions compare?
- ($\bigstar$) What are the significant properties of orthogonal matrices? What is special about their behavior as linear transformations?
- What is the relation between least-squares problems and projections?