Midterm 2: Review Problems

Tuesday, October 25

1 Computations

1. If $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$.
2. If $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$, find the coordinate vector of $p(t) = 1 + 3t - 6t^2$ relative to $\mathcal{B}$.
3. If the null space of a 5×4 matrix A is 2-dimensional, what is the dimension of the row space of A?
4. If A is a 7×5 matrix, what is the largest possible rank of A?
5. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for a vector space V such that $\mathbf{b}_1 = 2\mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3$, $\mathbf{b}_2 = 3\mathbf{c}_2 + \mathbf{c}_3$, and $\mathbf{b}_3 = -3\mathbf{c}_1 + 2\mathbf{c}_3$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. If $\mathbf{x} = \mathbf{b}_1 - 2\mathbf{b}_2 + 2\mathbf{b}_3$, find $[\mathbf{x}]_{\mathcal{C}}$.
6. If $A^2 - A = I$, what can you conclude about the eigenvalues of A?
7. Diagonalize the matrix $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ or show that it is not possible to do so.
8. Find the solution to $\min_x \|A\mathbf{x} - \mathbf{b}\|^2$, where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix},$$

   using both the normal equations and the QR factorization of A.
9. If $\langle , \rangle$ is an inner product on $\mathbb{P}_2$ defined by $\langle p, q \rangle = p(0)q(0) + 2p(1)q(1) + p(2)q(2)$, find an orthogonal basis for $\mathbb{P}_2$ with respect to $\langle , \rangle$.

2 True/False

For each statement, explain why it is true or give a counterexample.

1. If $\mathbf{x}$ is in V and if $\mathcal{B}$ contains n vectors, then $[\mathbf{x}]_{\mathcal{B}}$ is in $\mathbb{R}^n$.
2. The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic.
3. If H is a subspace of V then the dimension of H must be less than the dimension of V.
4. If B is any echelon form of A then the pivot columns of B form a basis for the column space of A.
5. The row space of A^T is the same as the column space of A.
6. If A and B are similar and A is diagonalizable, then B is also diagonalizable.
7. If E is an elementary matrix then the eigenvalues of EA are the same as the eigenvalues of A.
8. If an $n \times n$ matrix has n distinct eigenvalues then it has a basis of eigenvectors.
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10. If λ is an eigenvalue of A then it is also an eigenvalue of A^2.
11. If the columns of A are linearly independent then the equation $A\mathbf{x} = \mathbf{b}$ has exactly one least-squares solution.
12. The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the row space of A closest to $\mathbf{b}$.
13. If $\langle p(t), q(t) \rangle = p(0)q(1) + p(1)q(0)$, then $\langle , \rangle$ defines an inner product on $\mathbb{P}_1$.

3 Proofs

1. Show that if $C[a, b]$ is the set of all continuous functions on the interval $[a, b]$ then $C[a, b]$ is infinite-dimensional.
2. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors then $\mathbf{u}\mathbf{v}^T$ has rank 1.
3. Show that the rank of a matrix product AB is at most the minimum of (rank(A), rank(B)).
4. Show that if A is diagonalizable then so is $A^2 - 3A + 2I$.
5. If A and B are both diagonalizable and every eigenvector of A is an eigenvector of B (and vice versa), then $AB = BA$.
6. Show that if A is similar to B and B is similar to C then A is similar to C.
7. If $\langle p(t), q(t) \rangle = p(0)q(0) + 3p(1)q(1) - p(2)q(2)$, show that $\langle , \rangle$ does **not** define an inner product on $\mathbb{P}_2$.