Midterm 2: Review Problems

Tuesday, October 25

1 Computations

1. If $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$.

   ANSWER: $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & -3 \\ -2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

2. If $\mathcal{B} = \{1 - t^2, t - t^2, 2 - t + t^2\}$, find the coordinate vector of $p(t) = 1 + 3t - 6t^2$ relative to $\mathcal{B}$.

   ANSWER: Under the basis $\{1, t, t^2\}$, we get the answer

$$[p]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}.$$

3. If the null space of a 5×4 matrix A is 2-dimensional, what is the dimension of the row space of A?

   ANSWER: The Rank Theorem implies that the rank of A is $4 - 2$, or 2. Thus the dimension of the row space is also 2.

4. If A is a 7×5 matrix, what is the largest possible rank of A?

   ANSWER: The largest possible rank is 5. One example of such a matrix is a 5-by-5 identiy matrix with two rows of zeros beneath it.

5. Let $\mathcal{B}$ and $\mathcal{C}$ be bases for a vector space V such that $\mathbf{b}_1 = 2\mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3$, $\mathbf{b}_2 = 3\mathbf{c}_2 + \mathbf{c}_3$, and $\mathbf{b}_3 = -3\mathbf{c}_1 + 2\mathbf{c}_3$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. If $\mathbf{x} = \mathbf{b}_1 - 2\mathbf{b}_2 + 2\mathbf{b}_3$, find $[\mathbf{x}]_{\mathcal{C}}$.

   ANSWER:

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}; [\mathbf{x}]_{\mathcal{C}} = P[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}.$$

6. If $A^2 - A = I$, what can you conclude about the eigenvalues of A?

   ANSWER (updated): Suppose that v is an eigenvalue of A with eigenvalue λ. Then $(A^2 - A - I)v = 0$ since $A^2 - A - I = 0$, but we can also conclude that $(A^2 - A - I)v = A^2v - Av - v = \lambda^2 v - \lambda v - v = (\lambda^2 - \lambda - 1)v$. Therefore, $\lambda^2 - \lambda - 1 = 0$, so the only possible eigenvalues of A are $\lambda = \frac{1 \pm \sqrt{5}}{2}$.

7. Diagonalize the matrix $\begin{bmatrix} 4 & 0 & -1 \\ 0 & 4 & -1 \\ 1 & 0 & 2 \end{bmatrix}$ or show that it is not possible to do so.

   ANSWER: First find the characteristic polynomial of A: $\det(A - \lambda I) = (4 - \lambda)(\lambda - 3)^2$, so A has eigenvalues 4 with multiplicity 1 and 3 with multiplicity 2.

   Then find the dimensions of the eigenspaces. We know that the eigenspace with $\lambda = 4$ must have dimension 1, so just check the case $\lambda = 3$:

$$A - 3I = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Since the first and second columns of this matrix are linearly independent (as are the first and second rows), this matrix has rank at least 2. The null space therefore is at most 1-dimensional, so the eigenspace only has dimension 1. The matrix is not diagonalizable.

8. Find the solution to $\min_x \|A\mathbf{x} - \mathbf{b}\|^2$, where

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix},$$

using both the normal equations and the QR factorization of A.

ANSWER: Using the normal equations gives that

$$A^T A = \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}, A^T\mathbf{b} = \begin{bmatrix} 12 \\ 12 \\ 20 \end{bmatrix}, (A^T A)^{-1} A^T b = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$$

Orthogonalizing the second and third columns against the first give the matrix

$$\begin{bmatrix} 1 & 1 & 2.5 \\ 1 & -1 & -2.5 \\ 1 & -1 & -.5 \\ 1 & 1 & .5 \end{bmatrix},$$

after which orthogonalizing the third column against the second and then normalizing all columns gives the matrix

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

Then taking $Q^T A$ gives the factor $R = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, and finding $R^{-1}Q^T\mathbf{b}$ gives the same solution as before.

9. If $\langle , \rangle$ is an inner product on $\mathbb{P}_2$ defined by $\langle p, q\rangle = p(0)q(0) + 2p(1)q(1) + p(2)q(2)$, find an orthogonal basis for $\mathbb{P}_2$ with respect to $\langle , \rangle$.

ANSWER: $\langle 1, 1\rangle = 4$ and $\langle 1, t\rangle = 4$, so $\langle 1, t-1\rangle = 0$.

As a slight shortcut, $t-1$ has odd symmetry around the point 1 and $(t-1)^2$ has even symmetry around the same point, so $\langle t-1, (t-1)^2\rangle = 0$ since the inner product is symmetric with respect to 1. But $\langle 1, (t-1)^2\rangle = 2$, so subtract 1/2 to get $\langle 1, (t-1)^2 - 1/2\rangle = 0$.

The basis $\{1, t-1, (t-1)^2 - 1/2\rangle$ is therefore orthogonal with respect to this inner product.

2 True/False

For each statement, explain why it is true or give a counterexample.

1. If $\mathbf{x}$ is in V and if $\mathcal{B}$ contains n vectors, then $[\mathbf{x}]_{\mathcal{B}}$ is in $\mathbb{R}^n$. TRUE.

2. The vector spaces $\mathbb{P}_3$ and $\mathbb{R}^3$ are isomorphic. FALSE: $\mathbb{P}_3$ is 4-dimensional but $\mathbb{R}^3$ is only 3-dimensional.

3. If H is a subspace of V then the dimension of H must be less than the dimension of V. FALSE: V is itself a subspace of V, so the dimensions can be equal. Aside from this one exception (assuming finite-dimensional spaces), the statement is true.

4. If B is any echelon form of A then the pivot columns of B form a basis for the column space of A. FALSE: the corresponding columns of A form a basis for the column space of A, but the column space of B is not in general the same as the column space of A.

5. The row space of A^T is the same as the column space of A. TRUE.

6. If A and B are similar and A is diagonalizable, then B is also diagonalizable. TRUE: If $A = PDP^{-1}$ then $B = QAQ^{-1} = QPDP^{-1}Q^{-1} = (QP)D(QP)^{-1}$.

7. If E is an elementary matrix then the eigenvalues of EA are the same as the eigenvalues of A. FALSE, in general. For example, swapping the rows of the identity matrix will change its eigenvalues.

8. If an $n \times n$ matrix has n distinct eigenvalues then it has a basis of eigenvectors. TRUE, since eigenvectors corresponding to distinct eigenvalues are linearly independent.

9. If an $n \times n$ matrix has a basis of eigenvectors then it has n distinct eigenvalues. FALSE: for example, the identity matrix has 1 as its only eigenvalue but is diagonalizable (in fact, diagonal!)

10. If λ is an eigenvalue of A then it is also an eigenvalue of A^2. FALSE: λ^2 will be an eigenvalue of A^2. The corresponding eigenvectors, however, will be the same.

11. If the columns of A are linearly independent then the equation $A\mathbf{x} = \mathbf{b}$ has exactly one least-squares solution. TRUE, since A^TA will be invertible (or since the R factor in $A = QR$ will be invertible).

12. The least-squares solution of $A\mathbf{x} = \mathbf{b}$ is the point in the row space of A closest to $\mathbf{b}$. FALSE: it's the closest point in the column space, not the row space.

13. If $\langle p(t), q(t)\rangle = p(0)q(1) + p(1)q(0)$, then $\langle , \rangle$ defines an inner product on $\mathbb{P}_1$. FALSE, since it is not positive definite: $\langle t-1, t-1\rangle = 0$ even though $t - 1 \neq 0$.

3 Proofs

1. Show that if $C[a, b]$ is the set of all continuous functions on the interval $[a, b]$ then $C[a, b]$ is infinite-dimensional.

   ANSWER: The set of all polynomials on $[a, b]$ is infinite dimensional and a subspace of $C[a, b]$.

2. Show that if $\mathbf{u}$ and $\mathbf{v}$ are vectors then $\mathbf{u}\mathbf{v}^T$ has rank 1.

   ANSWER: $\mathbf{u}\mathbf{v}^T\mathbf{x} = \mathbf{u}(\mathbf{v}^T\mathbf{x})$, so the column space of $\mathbf{u}\mathbf{v}^T$ is spanned by the single vector $\mathbf{u}$.

3. Show that the rank of a matrix product AB is at most the minimum of (rank(A), rank(B)).

   ANSWER: The null space of AB contains the null space of B since if $B\mathbf{x} = \mathbf{0}$ then $AB\mathbf{x} = A\mathbf{0} = \mathbf{0}$. Since AB and B have the same number of columns, the rank of AB is therefore at most the rank of B.

   Then use the same logic on $(AB)^T = B^TA^T$ to show that the rank of $(AB)^T$ is at most the rank of A^T, which implies that the rank of AB is at most the rank of A.

4. Show that if A is diagonalizable then so is $A^2 - 3A + 2I$.

   ANSWER: If $A = PDP^{-1}$ then $A^2 - 3A + 2I = (PDP^{-1})^2 - 3PDP^{-1} + 2PIP^{-1} = P(D^2 - 3D + 2I)P^{-1}$.

5. If A and B are both diagonalizable and every eigenvector of A is an eigenvector of B (and vice versa), then $AB = BA$.

   ANSWER: If A and B have all the same eigenvectors then they are simultaneously diagonalizable, so $AB = (PD_AP^{-1})(PD_BP^{-1}) = PD_AD_BP^{-1} = PD_BD_AP^{-1} = (PD_BP^{-1})(PD_AP^{-1})$. This proof uses the fact that diagonal matrices commute.

6. Show that if A is similar to B and B is similar to C then A is similar to C.

   ANSWER: If $A = QBQ^{-1}$ and $B = PCP^{-1}$ then $A = Q(PCP^{-1})Q^{-1} = (QP)C(QP)^{-1}$.

7. If $\langle p(t), q(t)\rangle = p(0)q(0) + 3p(1)q(1) - p(2)q(2)$, show that $\langle , \rangle$ does **not** define an inner product on $\mathbb{P}_2$.

   ANSWER: It is not positive definite. Take the polynomial $t(t-1)$, which takes on the value 0 at 0 and 1 and 2 at 2. Then $\langle t(t-1), t(t-1)\rangle = 0 + 0 - 2^2 = -4$, but $\langle p, p\rangle$ must always be non-negative for $\langle , \rangle$ to define an inner product.