Midterm 1: Review Problems

Tuesday, September 20

1 Computations

1. If $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find all solutions to $A\mathbf{x} = \mathbf{b}$.
2. If $A = \begin{bmatrix} 1 & -3 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find all solutions to $A\mathbf{x} = \mathbf{b}$ in parametric vector form.
3. With the A given above, find all solutions to $A\mathbf{x} = \mathbf{0}$.
4. If $A = \begin{bmatrix} 2 & 4 \\ 1 & \alpha \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \beta \\ 5 \end{bmatrix}$, for what values of α and β will the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions?
5. If $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & h & 1 \\ 2 & 1 & h \end{bmatrix}$, find all values of h for which B is singular.
6. Show that $\begin{bmatrix} 1 \\ 11 \\ 17 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$
7. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$
8. Find the determinant of
$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
9. If $A = \begin{bmatrix} 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$, find a subset of columns of A that form a basis for the column space of A.

2 True/False

For each statement, explain why it is true or give a counterexample.

1. If S is a set of linearly dependent vectors then each vector of S is a linear combination of the other vectors in S.
2. If $B = A^{-1}$ then $AB = I$ and $BA = I$.
3. If the columns of an $n \times n$ matrix A are linearly dependent then $\det(A) = 0$.
4. If A is $m \times n$ and $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for any $\mathbf{b} \in \mathbb{R}^m$.
5. If a set in $\mathbb{R}^n$ is linearly dependent then the set contains more than n vectors.
6. The columns of any 4×5 matrix are linearly dependent.

7. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent but $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in $\text{Span}\{\mathbf{x}, \mathbf{y}\}$.
8. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ then $\{3\mathbf{u}, \mathbf{u} + 2\mathbf{v} + \mathbf{w}, \mathbf{w}\}$ is also a basis.
9. If A and B are invertible then $A + B$ and AB are also invertible.
10. If A is invertible then $\det(A^{-1}) = 1/\det(A)$.

3 Proofs

1. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be vectors in $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\text{Span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$. Show that $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are in the span of $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$.
2. If the range space of an $n \times n$ matrix A is $\mathbb{R}^n$ for $n > 0$, show that A must be invertible.
3. Determine, with proof, whether the transformation $T(x, y) = (x - 2y, x + 3, 2x - 5y)$ is linear.
4. Let S be the set of all real-valued functions f such that $f' = f$. Determine whether S is a subspace.
5. Let W be the union of the first and third quadrants in the xy-plane, so $W = \{(x, y) : xy \geq 0\}$. Determine whether W is a subspace.
6. Let A, B, C be matrices such that A and C are invertible and $A^{-1} = C^{-1}B$. Show that B is also invertible.
7. Prove that if S and T are subspaces of a vector space V then $S \cap T$ is also a subspace.
8. Let T be a linear transformation such that $T(\mathbf{v}_1) = \mathbf{u}_1$ and $T(\mathbf{v}_2) = \mathbf{u}_2$. If $\mathbf{w} = 3\mathbf{u}_1 - 2\mathbf{u}_2$, show that there exists $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{w}$.
9. Show that if the columns of an $n \times n$ matrix A are linearly independent then the columns of A^2 span $\mathbb{R}^n$.