Midterm 1: Review Problems

Tuesday, September 20

1 Computations

1. If $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find all solutions to $A\mathbf{x} = \mathbf{b}$.

   ANSWER: It's most straightforward to row reduce the augmented matrix.

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 4 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 0 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 0 \end{bmatrix} \end{aligned}$$

   so x_3 is free, $x_2 = -3x_3$, and $x_1 = 1 + 2x_3$. In parametric vector form, the set of solutions is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

2. If $A = \begin{bmatrix} 1 & -3 & -1 \\ 3 & -7 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find all solutions to $A\mathbf{x} = \mathbf{b}$ in parametric vector form.

   ANSWER: Same strategy as before.

$$\begin{aligned} \begin{bmatrix} 1 & -3 & -1 & 1 \\ 3 & -7 & 1 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & -3 & -1 & 1 & \\ 0 & 2 & & 4 & -2 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -3 & -1 & 1 & \\ 0 & 1 & & 2 & -1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 5 & -2 & \\ 0 & 1 & & 2 & -1 \end{bmatrix} \end{aligned}$$

   So the solution set is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

3. With the A given above, find all solutions to $A\mathbf{x} = \mathbf{0}$.

   ANSWER: The set of solutions to $A\mathbf{x} = \mathbf{0}$ is the set of all $\mathbf{v} - \mathbf{w}$, where $\mathbf{v}$ and $\mathbf{w}$ are solutions to $A\mathbf{x} = \mathbf{b}$ for any $\mathbf{b}$. The set of solutions is therefore $\left\{ t \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}$

4. If $A = \begin{bmatrix} 2 & 4 \\ 1 & \alpha \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} \beta \\ 5 \end{bmatrix}$, for what values of α and β will the system $A\mathbf{x} = \mathbf{b}$ have infinitely many solutions?

ANSWER: For a square system to have infinitely many solutions the columns of A must be linearly dependent, so $\alpha = 2$. The system must also be consistent, so $\mathbf{b}$ is in the span of the columns of A. But since both columns of A are spanned by a single column of A, this means that $\mathbf{b}$ must be a multiple of the first (or equivalently, second) columns of A. Therefore $\beta = 10$.

5. If $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & h & 1 \\ 2 & 1 & h \end{bmatrix}$, find all values of h for which B is singular.

   ANSWER:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ 1 & h & 1 \\ 2 & 1 & h \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 0 \\ 1 & h & 0 \\ 2 & 1 & h-2 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 \\ 1 & h-1 & 0 \\ 2 & -1 & h-2 \end{bmatrix} = (h-1)(h-2),$$

   so B is singular if and only if $h = 1$ or $h = 2$.

6. Show that $\begin{bmatrix} 1 \\ 11 \\ 17 \end{bmatrix}$ is in Span $\left\{ \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$

   ANSWER: The most straightforward way is to take this problem as equivalent to determining whether the system $[\mathbf{u}\,\mathbf{v}]\mathbf{x} = \mathbf{w}$ is consistent. You can solve it by row reducing the augmented system, but in any case the particular linear combination that works is

$$\begin{bmatrix} 1 \\ 11 \\ 17 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

7. Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

   ANSWER: You could do this using the theorem that showed up alongside Cramer's Rule, but I don't like Cramer's Rule so I'll use row reduction (i.e. multiply by a series of row elementary matrices) instead:

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}$$

Therefore, $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$.

(If you want to be safe, you should check that $AA^{-1} = I$)

8. Find the determinant of

$$\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

ANSWER: Row reduction is probably the way to go here.

$$\det \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$
$$= \det \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 4 & 4 \end{bmatrix}$$
$$= \det \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$
$$= 8.$$

9. If $A = \begin{bmatrix} 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$, find a subset of columns of A that form a basis for the column space of A.

   ANSWER: Note that A is already in echelon form, so since it has three row pivots the column space is $\mathbb{R}^3$. The column vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ are linearly independent and span this space (you can check that the determinant of the matrix formed by those three vectors is nonzero), and therefore form a basis.

2 True/False

For each statement, explain why it is true or give a counterexample.

1. If S is a set of linearly dependent vectors then each vector of S is a linear combination of the other vectors in S.

   FALSE. Counterexample with two vectors: $\mathbf{u} = \mathbf{0}$ but $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{v}$ and $\mathbf{u}$ are linearly independent but $\mathbf{v}$ is not a multiple of $\mathbf{u}$. Lesson: just because *some* vectors are redundant doesn't mean all of them are.

2. If $B = A^{-1}$ then $AB = I$ and $BA = I$. TRUE by definition

3. If the columns of an $n \times n$ matrix A are linearly dependent then $\det(A) = 0$. TRUE, Invertible Matrix Theorem

4. If A is $m \times n$ and $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for any $\mathbf{b} \in \mathbb{R}^m$. FALSE. $A\mathbf{x} = \mathbf{b}$ will have either zero solutions or infinitely many.

5. If a set in $\mathbb{R}^n$ is linearly dependent then the set contains more than n vectors. FALSE. Even if $n = 1$, the set $\{\mathbf{0}\}$ is linearly dependent without containing more than n vectors.

6. The columns of any 4×5 matrix are linearly dependent. TRUE, since they correspond to 5 vectors in $\mathbb{R}^4$.

7. If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent but $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent, then $\mathbf{z}$ is in $\text{Span}\{\mathbf{x}, \mathbf{y}\}$. TRUE.

8. If $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for $\mathbb{R}^3$ then $\{3\mathbf{u}, \mathbf{u} + 2\mathbf{v} + \mathbf{w}, \mathbf{w}\}$ is also a basis. TRUE. One argument is that

   $$[3\mathbf{u}, \mathbf{u} + 2\mathbf{v} + \mathbf{w}, \mathbf{w}] = [\mathbf{u}\,\mathbf{v}, \mathbf{w}] \begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} := [\mathbf{u}\,\mathbf{v}, \mathbf{w}]B,$$

   where B is invertible. So if $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ is invertible (i.e. the column vectors form a basis) then so is $[3\mathbf{u}, \mathbf{u} + 2\mathbf{v} + \mathbf{w}, \mathbf{w}]$.

9. If A and B are invertible then $A + B$ and AB are also invertible.

   FALSE: AB is invertible but $A + B$ might not be. If $B = -A$ then $A + B = 0$, which is certainly not invertible.

10. If A is invertible then $\det(A^{-1}) = 1/\det(A)$. TRUE, since $1 = \det I = \det(AA^{-1}) = \det A \det A^{-1}$.

3 Proofs

1. Let $\mathbf{u}_1, \ldots, \mathbf{u}_m$ be vectors in $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\text{Span}(\mathbf{w}_1, \ldots, \mathbf{w}_n)$. Show that $\mathbf{u}_1, \ldots, \mathbf{u}_m$ are in the span of $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$.

   ANSWER: This is obnoxious to write in vector form, so rephrase it as a matrix problem instead. This is equivalent to showing that if $U = VX$ and $V = WY$ then $U = WZ$ for some Z, where U, V, W have $\mathbf{u}_i, \mathbf{v}_i, \mathbf{w}_i$ as column vectors.

   It then follows from the givens that $U = VX = (WY)X = W(YX)$. This completes the proof.

2. If the range space of an $n \times n$ matrix A is $\mathbb{R}^n$ for $n > 0$, show that A must be invertible.

   ANSWER: This follows from the Invertible Matrix Theorem. . . . I think that might be all you had to say?

   If not, then $A\mathbf{x} = \mathbf{e}_i$ has a solution $\mathbf{b}_i$ for each $1 \leq i \leq n$. It follows that $AB = A[\mathbf{b}_1, \ldots, \mathbf{b}_n] = [\mathbf{e}_1, \ldots, \mathbf{e}_n] = I$, and since A is $n \times n$, A is invertible.

3. Determine, with proof, whether the transformation $T(x, y) = (x - 2y, x + 3, 2x - 5y)$ is linear.

   ANSWER: Excuse the confusing typo in the original problem ("x3" should be "x + 3"). The easiest way to handle this is to note that $T(0, 0) = (0, 3, 0)$, but $T(0, 0) + T(0, 0) \neq T[(0, 0) + (0, 0)]$.

   In general, T must take $\mathbf{0}$ to $\mathbf{0}$ to be a linear transformation.

4. Let S be the set of all real-valued functions f such that $f' = f$. Determine whether S is a subspace.

   ANSWER: check that it satisfies the two definitions.

   If $f, g \in S$, then $(f + g)' = f' + g' = f + g$, so $f + g \in S$.

   If $f \in S$, then $(cf)' = cf' = cf$, so $cf \in S$ for any $c \in \mathbb{R}$.

   S is therefore a subspace.

5. Let W be the union of the first and third quadrants in the xy-plane, so $W = \{(x, y) : xy \geq 0\}$. Determine whether W is a subspace.

   ANSWER: it is straightforward to check that if $(x, y) \in W$ then $c(x, y) \in W$, but the property of closure under addition is not so clear. In fact, it is false. Try $(5, 3)$ and $(-3, -5)$. Both are in W but the sum is $(2, -2)$, which is not in W. W is therefore not a subspace.

6. Let A, B, C be matrices such that A and C are invertible and $A^{-1} = C^{-1}B$. Show that B is also invertible.

   ANSWER: $I = AA^{-1} = (AC^{-1})B$, so B is invertible and $B^{-1} = AC^{-1}$.

   Alternately, $CA^{-1} = CC^{-1}B = B$, and we know that C and A^{-1} are invertible so their product is also invertible.

7. Prove that if S and T are subspaces of a vector space V then $S \cap T$ is also a subspace.

   ANSWER: Take any $\mathbf{u}, \mathbf{v} \in S \cap T$. Then $u, v \in S$ and $u, v \in T$. This means that for any $c \in \mathbb{R}$, $c\mathbf{u} + \mathbf{v} \in S$ and $c\mathbf{u} + \mathbf{v} \in T$ (since S and T are subspaces), so it follows that $c\mathbf{u} + \mathbf{v} \in S \cap T$. Therefore, $S \cap T$ is a subspace.

8. Let T be a linear transformation such that $T(\mathbf{v}_1) = \mathbf{u}_1$ and $T(\mathbf{v}_2) = \mathbf{u}_2$. If $\mathbf{w} = 3\mathbf{u}_1 - 2\mathbf{u}_2$, show that there exists $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{w}$.

   ANSWER: $T(3\mathbf{v}_1 - 2\mathbf{v}_2) = T(3\mathbf{v}_1) + T(-2\mathbf{v}_2) = 3T(\mathbf{v}_1) - 2T(\mathbf{v}_2) = 3\mathbf{u}_1 - 2\mathbf{u}_2 = \mathbf{w}$.

9. Show that if the columns of an $n \times n$ matrix A are linearly independent then the columns of A^2 span $\mathbb{R}^n$.

   ANSWER: Since the columns of A are linearly independent and A is $n \times n$, A is invertible. Therefore A^2 is also invertible, and by the Invertible Matrix Theorem its columns span $\mathbb{R}^n$.