Final Review Problems

Friday, December 2

1. Consider the second-order differential equation $y'' + \alpha y' + 9y = 0$. For what values of α will the auxiliary equation have real roots? Complex? Multiple? Find the general solution in all three cases.

   ANSWER: The discriminant of the auxiliary equation is $\alpha^2 - 36$, so the equation will have distinct real roots when $|\alpha| > 6$, a multiple root when $\alpha = \pm 6$, and complex roots when $|\alpha| < 6$.

   In the real distinct case, set $r_1, r_2 = \frac{-\alpha \pm \sqrt{\alpha^2 - 36}}{2}$, and the general solution is $y(t) = c_1 e^{-r_1 t} + c_2 e^{-r_2 t}$.

   In the second case, the multiple root is $r = 3$ when $\alpha = -6$ or $r = -3$ when $\alpha = 6$. Then the general solution is $y(t) = c_1 t e^{rt} + c_2 e^{rt}$.

   In the third case, set $a + ib = -\frac{\alpha}{2} \pm i\frac{\sqrt{36 - \alpha^2}}{2}$. Then the general solution is $y(t) = c_1 e^{at}\cos bt + c_2 e^{at}\sin bt$.

2. Find the general solution to the differential equation $y'' - y' - 2y = e^{-t}$.

   ANSWER: The roots of the auxiliary equation are $r = -1, r = 2$, so the general solution to the homogeneous equation is of the form $y = c_1 e^{-t} + c_2 e^{2t}$. Note that the right hand side is a solution to the homogeneous equation, so we guess a nonhomogeneous solution of the form $y = cte^{-t}$. Plugging this form into the equation $y'' - y' - 2y = e^{-t}$ gives the solution $c = -1/3$, so $y_p = -\frac{1}{3}te^{-t}$.

   Thee general solution is therefore $y = c_1 e^{-t} + c_2 e^{2t} - \frac{1}{3}te^{-t}$.

3. Use the method of undetermined coefficients to find a general solution to the system

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \mathbf{x}(t) + e^{-2t} \begin{bmatrix} t \\ 3 \end{bmatrix}.$$

   ANSWER: The eigenvalues of A are $\lambda = 1$ with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\lambda = 2$ with eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so a fundamental matrix is $X(t) = \begin{bmatrix} e^t & e^{2t} \\ 0 & e^{2t} \end{bmatrix}$.

   Then for the particular solution, guess a solution of the form $\mathbf{x}(t) = \mathbf{a}te^{-2t} + \mathbf{b}e^{-2t}$, with $\mathbf{x}'(t) = -2\mathbf{a}te^{-2t} + (\mathbf{a} - 2\mathbf{b})e^{-2t}$. Equating terms corresponding to different functions gives the two sets of equations

$$-2\mathbf{a} = A\mathbf{a} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\mathbf{a} - 2\mathbf{b} = A\mathbf{b} + \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

   Solving the first system gives $\mathbf{a} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$, then solving the second gives $\mathbf{b} = \begin{bmatrix} 5/36 \\ -3/4 \end{bmatrix}$.

4. Use variation of parameters to find a general solution of the system

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

   ANSWER: The eigenvalues are $\pm i$, so a fundamental matrix is $X(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$.

   Then

$$
\begin{aligned}
X(t)\int X(s)\inf f &= X(t)\int \begin{bmatrix}\cos s & -\sin s\\ \sin s & \cos s\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\,ds\\
&= X(t)\int \begin{bmatrix}\cos s\\ \sin s\end{bmatrix}\,ds\\
&= \begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\begin{bmatrix}\sin t\\ -\cos t\end{bmatrix} + X(t)\mathbf{c}\\
&= X(t)\mathbf{c} + \begin{bmatrix}0\\ -1\end{bmatrix}.
\end{aligned}
$$

5. Find e^{At} where $A = \begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}$.

   Finding the eigenvalues and eigenvectors of A gives a possible diagonaliziation $A = VDV^{-1}$ where $D = \begin{bmatrix}-1 & 0\\ 0 & 3\end{bmatrix}$ and $V = \begin{bmatrix}1 & 1\\ -2 & 2\end{bmatrix}$.

   Then

$$
\begin{aligned}
e^{At} &= Ve^{Dt}V^{-1}\\
&= \frac{1}{4}\begin{bmatrix}e^{-t} & e^{3t}\\ -2e^{-t} & 2e^{3t}\end{bmatrix}\begin{bmatrix}2 & -1\\ 2 & 1\end{bmatrix}\\
&= \frac{1}{4}\begin{bmatrix}2e^{-t}+2e^{3t} & -e^{-t}+e^{3t}\\ -4e^{-t}+4e^{3t} & 2e^{-t}+2e^{3t}\end{bmatrix}.
\end{aligned}
$$

6. A metal rod of length L starts at 100 degrees Celsius when suddenly the temperature at both endpoints is fixed to zero. If the rate of diffusion of heat through the rod is governed by the equation $u_t = u_{xx}$, find the temperature of the rod at any given point x and time t.

   ANSWER: The temperature of the rod will be of the form

$$
u(x,t) = \sum_{n=1}^{\infty} c_n e^{-(n\pi/L)^2 t}\sin\frac{n\pi x}{L},
$$

   where the Fourier series for the initial temperature is given by

$$
\begin{aligned}
100 &= \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\\
c_n &= \frac{2}{L}\cdot 100\cdot\int_0^L \sin\frac{n\pi x}{L}\,dx\\
&= \frac{200}{n\pi}(1-\cos n\pi)\\
&= \begin{cases}0 & n = 2k\\ \frac{400}{\pi n} & n = 2k+1\end{cases}.
\end{aligned}
$$

7. Find a formal solution to the vibrating string problem governed by the given set of equations:

$$
\begin{aligned}
u_{tt} &= 9u_{xx}, 0 < x < \pi, t > 0\\
u(0,t) &= u(\pi,t) = 0, t > 0\\
u(x,0) &= \sin 4x + 7\sin 5x, 0 < x < \pi\\
u_t(x,0) &= 0.
\end{aligned}
$$

ANSWER: Since the string has no starting velocity, the answer will be of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos n\alpha t \sin nx,$$

where

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx.$$

Based on the value of f given, the exact Fourier series is given by $a_4 = 1, a_5 = 7$. Since $\alpha = 3$, the answer is therefore

$$u(x,t) = \cos 12t \sin 4x + 7 \cos 15t \sin 5x.$$