Solutions to Online Practice Final

Friday, December 10

Problem 1

1. Find bases for the column and null spaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

ANSWER: Since A is already in echelon form with nonzero elements in the pivot positions, A has full rank. Thus the column space is $\mathbb{R}^3$, and any basis for $\mathbb{R}^3$ will be an acceptable answer (in particular, the first three columns of A will do).

By the Rank Nullity Theorem, the null space of A is 1-dimensional. Solving $Ax = 0$ gives that $Null(A) = \text{Span}\left\{\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right\}$.

2. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

Find the eigenvalues and corresponding eigenvectors of A, and diagonalize A.

ANSWER: Since all of the rows of A are multiples of the first one, A has rank one and can therefore be factorized as $A = vv^T$ (here, $v = (1, 1, 2)^T$). Then v is the only nonzero eigenvector of A, and since $Av = 6v$ we conclude that $\lambda_1 = 6$. Then the other eigenvalue of A is $\lambda_2 = 0$ with a two-dimensional eigenspace. This space will be spanned by any two linearly independent vectors orthogonal to v, so $(1, -1, 0)$ and $(0, -2, 1)$ will do.

This choice of basis for the zero eigenspace gives the diagonalization

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -2 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & -2 \\ 2 & 0 & 1 \end{pmatrix}^{-1}$$

Problem 2

Consider the parameterized differential equation

$$x'' + 4x' + \alpha x = 0,$$

where α is a real constant. Find all the roots of the auxiliary equation. For what values of α are they real, complex, or multiple? In each case find the general solution to the differential equation.
ANSWER: The roots are given by $r = \frac{-4\pm\sqrt{16-4\alpha}}{2} = -2 \pm \sqrt{4-\alpha}$, and we get the following three cases:

1. $\alpha < 4$: the roots are real, and the general solution is $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$.
2. $\alpha = 4$: there is a double root $r = -2$, and the general solution is $x(t) = c_1 t e^{-2t} + c_2 e^{-2t}$.
3. $\alpha > 4$: the roots are complex with roots $-2 \pm i\sqrt{\alpha - 4}$, and the general solution is given by $x(t) = c_1 e^{-2t} \sin t\sqrt{\alpha-4} + c_2 e^{-2t} \cos t\sqrt{\alpha-4}$.

Problem 3

Let $q_1, \dots, q_k$ be a set of orthonormal vectors in $\mathbb{R}^n$. Show that there exist vectors $q_{k+1}, \dots, q_n$ so that $Q := (q_1 \quad \dots \quad q_n)$ is an orthonormal matrix.
ANSWER: We can build this matrix Q over a series of steps: If $k = n$, then we are done. Otherwise, the set $\{q_1, \dots, q_k\}$ does not span $\mathbb{R}^n$ and we can find some $v_{k+1} \in \mathbb{R}^n$ not in the span of this set. Now v_{k+1} may not be orthogonal to the vectors $\{q_1, \dots, q_n\}$, but we can orthogonalize it using the Gram-Schmidt process and rescale so that is has norm 1. Now we have $k + 1$ orthogonal vectors, so repeat in this manner until we have n orthogonal vectors.

Problem 4

Consider the function $f(x) = \cos\left(\sqrt{2}x\right)$ on the interval $(0, \pi)$.

1. Extend it to an odd function on $(-\pi, \pi)$.

   ANSWER: $\tilde{f}(x) = \begin{cases} f(x) & x \in (0, \pi) \\ x & x = 0 \\ -f(-x) & x \in (-\pi, 0) \end{cases}$.

2. Compute the Fourier sine series of the function on the interval $0, \pi$.

   ANSWER:

$$
\begin{aligned}
c_n &= \frac{2}{\pi} \int_{x=0}^{\pi} \cos\left(\sqrt{2}x\right) \sin(nx)\, dx \\
&= \frac{2}{\pi} \int_{x=0}^{\pi} \frac{\sin\left((n+\sqrt{2})x\right) + \sin\left((n-\sqrt{2})x\right)}{2}\, dx \\
&= \frac{1}{\pi} \left[-\frac{1}{n+\sqrt{2}} \cos\left((n+\sqrt{2})x\right) - \frac{1}{n-\sqrt{2}} \cos\left((n-\sqrt{2})x\right) \right]_{x=0}^{\pi} \\
&= \frac{1}{\pi} \left[-\frac{1}{n+\sqrt{2}} - \frac{1}{n-\sqrt{2}} + \frac{1}{n+\sqrt{2}} \cos\left((n+\sqrt{2})\pi\right) + \frac{1}{n-\sqrt{2}} \cos\left((n-\sqrt{2})\pi\right) \right].
\end{aligned}
$$

   You might be able to simplify this a little further, but not much. At least note that this formula would not hold if the $\sqrt{2}$ term had been replaced with an integer. since for at least one term $\sin(n-k)x$ would be zero.

Problem 5

Find all solutions of the form $u(x,t) = X(x)T(t)$ for the following equations:

$$\begin{aligned} u_t(x,t) &= u_{xx}(x,t) \\ u(0,t) &= 0 \\ u_x(L,t) &= 0. \end{aligned}$$

Note that is the heat equation with one end insulated and the other end with temperature fixed to zero. Assuming that $u(x,t) = X(x)T(t)$ and plugging this form into the first equation gives

$$\begin{aligned} X(x)T'(t) &= X''(x)T(t) \\ \frac{T'(t)}{T(t)} &= \frac{X''(x)}{X(x)}. \end{aligned}$$

Since this is true for all times x and t, varying x while keeping t fixed (or vice versa) shows that both sides must be constants. Thus X and T satisfy

$$\begin{aligned} X''(x) &= \lambda X(x) \\ T'(t) &= \lambda T(t), \end{aligned}$$

and we can break the solution types into three cases depending on whether $\lambda > 0$, $\lambda = 0$, or $\lambda < 0$.

1. If $\lambda > 0$ then $T(t) = ce^{\lambda t}$ and $X(x) = c_1 e^{\sqrt{\lambda}t} + c_2 e^{-\sqrt{\lambda}t}$. But equations of this type will necessarily fail the boundary conditions: if $X(0) = 0$ then $c_1 + c_2 = 0$, but if $X'(L) = 0$ then $c_1\sqrt{\lambda}e^{\sqrt{\lambda}L} - c_2\sqrt{\lambda}e^{-\sqrt{\lambda}L} = 0$. This gives the 2-by-2 system

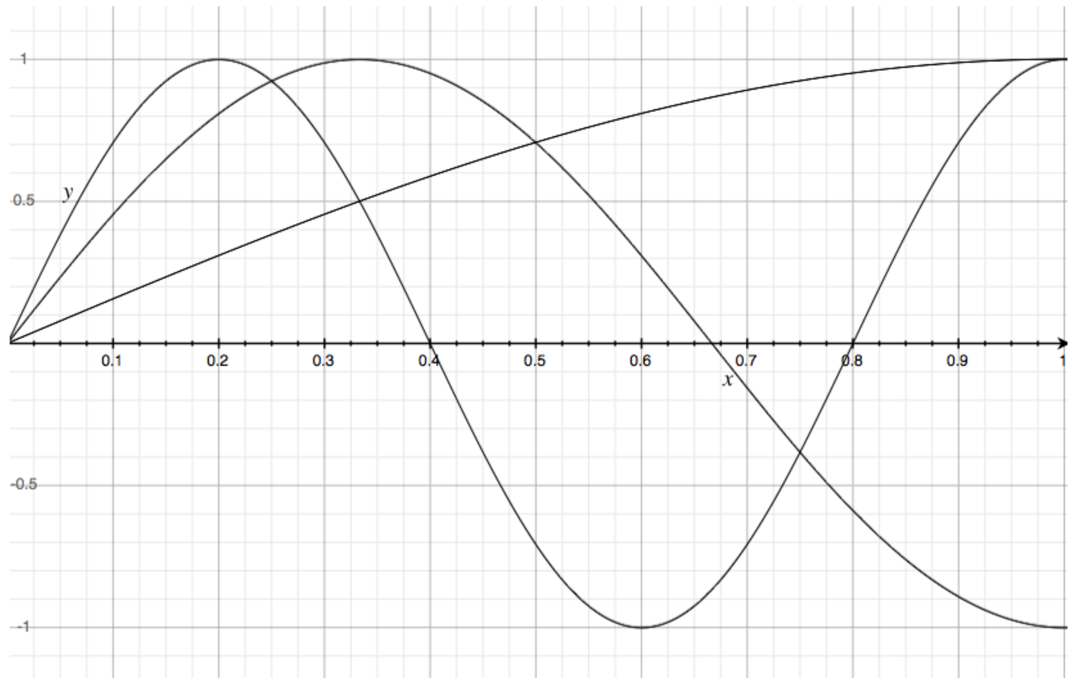
$$\begin{bmatrix} 1 & 1 \\ \sqrt{\lambda}e^{\sqrt{\lambda}L} & -\sqrt{\lambda}e^{-\sqrt{\lambda}L} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   Since the bottom right term is the only negative one this matrix is invertible, and so the system has only the trivial solution.

2. If $\lambda = 0$ then $T(t)$ is constant and $X(x) = ax + b$. The left boundary condition implies that $b = 0$ and the right boundary implies that $a = 0$, so we are once again left with only the trivial solution.

3. If $\lambda < 0$ then $T(t) = ce^{-\lambda t}$ and $X(x) = c_1 \sin(\sqrt{\lambda}t) + c_2 \cos\sin(\sqrt{\lambda}t)$. The left boundary condition implies that $c_2 = 0$ for any λ. The right boundary condition then implies that $c_1\sqrt{\lambda}\cos(\sqrt{\lambda}L) = 0$, which has a nontrivial solution if and only if $\sqrt{\lambda}L = \frac{\pi}{2} + n\pi$ for some integer $n \geq 0$.

Therefore, if $u(x,t) = X(x)T(t)$ and the boundary conditions are as given, then $u(x,t) = c\sin\left(\frac{(2n+1)\pi x}{2L}\right) e^{-\left(\frac{(2n+1)\pi}{2L}\right)^2 t}$ for some $n \geq 0$. The graph below shows these solutions for $L = 1$ and $n = 0, 1, 2$ at $t = 0$.


1
0.5
y
-0.5
-1
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x