Quiz 7; Tuesday, October 18
MATH 54 with Ming Gu
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Student name:

You have 15 minutes to complete the quiz. Calculators are not permitted.

1. (4 points) Find the QR factorization of the matrix $\begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$. In other words, write the matrix as a product QR, where Q is orthogonal and R is upper triangular.

   ANSWER: Define $\mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Then $\mathbf{a}_1$ is already a unit vector so we can say $\mathbf{q}_1 = \mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $r_{11} = \|\mathbf{a}_1\| = 1$.

   Then $\mathbf{q}_1^T \mathbf{a}_2 = -2$, so $r_{12} = -2$. $\mathbf{a}_2 - (-2)\mathbf{q}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\mathbf{q}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $r_{22} = 3$.

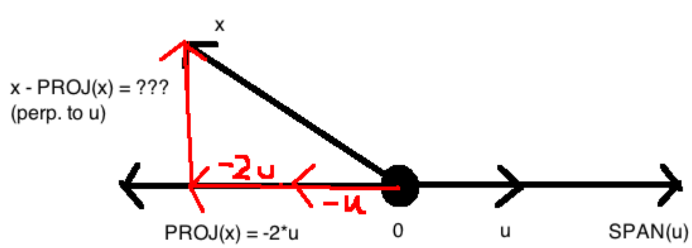
   Therefore,

$$\begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}.$$

   It is simple to confirm that Q is in fact orthogonal and R is upper triangular.

2. (4 points) Let $\mathbf{u}$ be a unit vector in $\mathbb{R}^2$ and let $\mathbf{x}$ be a vector such that $\mathbf{u}^T\mathbf{x} = -2$ and that $\mathbf{x}$ and $\mathbf{u}$ are linearly independent. Sketch and label $\mathbf{u}$, $\mathbf{x}$, $\mathrm{proj}_{\mathbf{u}}(\mathbf{x})$, and $\mathbf{x} - \mathrm{proj}_{\mathbf{u}}(\mathbf{x})$. There may be multiple correct sketches.

   ANSWER: A really good sketch:



   The important characteristics:

   (a) $\mathbf{x}$ and $\mathbf{u}$ are linearly independent, so $\mathbf{x}$ is not on the line spanned by $\mathbf{u}$.
   (b) $\mathbf{x}$ and $\mathbf{u}$ form an obtuse angle.
   (c) The projection of $\mathbf{x}$ onto $\mathbf{u}$ is twice as long as $\mathbf{u}$, in the opposite direction.
   (d) $x - \mathrm{proj}_{\mathbf{u}}(x)$ is perpendicular to $\mathbf{u}$.

3. (4 points) Mark each statement as True or False. You do not have to explain your reasoning.

(a) If L is a line through $\mathbf{0}$ and $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $\|\hat{\mathbf{y}}\|$ gives the distance from $\mathbf{y}$ to L. FALSE: $\|\mathbf{y} - \hat{\mathbf{y}}\|$ gives the distance.

(b) For any subspace W and vector $\mathbf{x}$, $\mathbf{x} - \operatorname{proj}_W \mathbf{x}$ *must* be an element of $W^\perp$. TRUE.

(c) For any matrix U, $UU^T\mathbf{y}$ is the projection of $\mathbf{y}$ onto the span of U. FALSE: U must be orthogonal for this to work in general.

(d) If Q is an orthogonal matrix then Q^{-1} is also an orthogonal matrix. TRUE: since $Q^{-1} = Q^T$, $(Q^{-1})^T Q^{-1} = (Q^T)^T Q^T = QQ^T = I$.