Quiz 6; Tuesday, October 11
MATH 54 with Ming Gu
GSI: Eric Hallman

Student name:

You have 15 minutes to complete the quiz. Calculators are not permitted.

1. (4 points) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. State which eigenvector is associated with which eigenvalue.

   ANSWER: the characteristic polynomial of the matrix is $(2-\lambda)(2-\lambda)-1 = \lambda^2-4\lambda+3 = (\lambda-3)(\lambda-1)$, so the eigenvalues are 1 and 3.

   An eigenvector with $\lambda = 1$ will be in the null space of $A - I$, which is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, so any scalar multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ will do.

   As for the eigenvector $\lambda = 3$, $A - 3I$ is equal to $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, so any scalar multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will be an eigenvector.

2. (2 points) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

   ANSWER: $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, perhaps different by scaling the columns of P or by swapping the columns of both P and D.

3. (6 points) Mark each statement as True or False. You do not have to explain your reasoning.

   (a) If A is similar to B then A^2 is similar to B^2. TRUE: If $PAP^{-1} = B$ then $B^2 = PAP^{-1}PAP^{-1} = PA^2P^{-1}$.

   (b) If A is similar to B then A and B have the same eigenvalues and eigenvectors. FALSE: they will have the same eigenvalues but will not in general have the same eigenvectors.

   (c) If $AP = PD$ where D is a diagonal matrix then the nonzero columns of P are eigenvectors of A. TRUE.

   (d) Elementary row operations do not change the eigenvalues of a matrix. FALSE. Multiplying a row by a scalar can easily change the eigenvalues of a matrix.

   (e) If a matrix is diagonalizable then its columns are linearly independent. FALSE: As a silly example, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable. More generally, if its columns are LI then the matrix is invertible, but a matrix does not need to be invertible in order to be diagonalizable.

   (f) Any eigenvector of A is also an eigenvector of A^2. TRUE: $A^2v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$.