Quiz 5; Tuesday, October 4
MATH 54 with Ming Gu
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Student name:

You have 15 minutes to complete the quiz. Calculators are not permitted.

1. (4 points) If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \end{bmatrix} \right\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\} = \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$ find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and from $\mathcal{C}$ to $\mathcal{B}$. Be sure to specify which is which. (Hint: once you have found one it should not be too hard to find the other.)

   ANSWER: The change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ is given by

   $$P_{\mathcal{C}\leftarrow\mathcal{B}} = C^{-1}B = \frac{1}{8}\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 4 & 8 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}.$$

   We can check this by observing that $\mathbf{b}_1 = 2\mathbf{c}_1$ and $\mathbf{b}_2 = 3\mathbf{c}_1 - \mathbf{c}_2$.

   Then

   $$P_{\mathcal{B}\leftarrow\mathcal{C}} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ 0 & 2 \end{bmatrix}.$$

2. (4 points) Given that $\lambda = 4$ is an eigenvalue of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$, find one corresponding eigenvector.

   ANSWER: We have to find an element in the null space of $A - 4I$, which we can do by reducing the matrix to echelon form:

   $$\begin{aligned} \begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix} - 4I &\sim \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

   An element of the null space (and therefore an eigenvector with eigenvalue 4) is $\mathbf{v} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. Since the null space has dimension 1, all other possible answers are parallel to this one.

3. (4 points) Mark each statement as True or False. You do not have to explain your reasoning.

(a) If the null space of an 8×7 matrix A is 5-dimensional, then the row space of A must be 3-dimensional. FALSE: it must be 2-dimensional.

(b) The row space of A^T is the same as the column space of A. TRUE.

(c) If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A. FALSE: we need the condition $\mathbf{x} \neq \mathbf{0}$ in order for this to be true; otherwise any value of λ would be an eigenvalue of A.

(d) If two matrices A and B are row equivalent then they have the same eigenvalues. FALSE: for example, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ has eigenvalues 0 and 1, but $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has eigenvalues 0 and 2.