Quiz 4; Tuesday, September 27
MATH 54 with Ming Gu
GSI: Eric Hallman

Student name:

You have 15 minutes to complete the quiz. Calculators are not permitted.

1. (4 points) If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{B}}$.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \frac{-1}{3} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

2. (4 points) Find bases for the null space **and** the column space of $\begin{bmatrix} 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Be sure to state which basis is which.

   The column space of A is all of $\mathbb{R}^3$ since A has three pivots, so any old basis will do. The standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the simplest.

   If we set up the equation $A\mathbf{x} = \mathbf{0}$ in terms of free variables we might get that x_2 and x_5 are free, so let the basis be determined by the two solutions that follow from $(x_2, x_5) = (1, 0)$ and $(x_2, x_5) = (0, 1)$: these are the vectors $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. These two vectors together make a basis for the null space.

   (We can also tell that the null space must be 2-dimensional because A acts on vectors in $\mathbb{R}^5$ and has 3 pivots, so 5-3=2.)

3. (4 points) Mark each statement as True or False. You do not have to explain your reasoning.
   (a) There are exactly three 2-dimensional subspaces of $\mathbb{R}^3$. FALSE: there are infinitely many.
   (b) Every 2-dimensional vector space is isomorphic to $\mathbb{R}^2$. TRUE.
   (c) If $T : V \to W$ is an isomorphism between n-dimensional vector spaces and $\{v_1, \dots, v_n\}$ are linearly independent in V, then $\{T(v_1), \dots, T(v_n)\}$ span W. TRUE, since the v_i form a basis and so the $T(v_i)$ must also form a basis.
   (d) The null space of A is equal to the column space of A^T. FALSE.