Quiz 2; Tuesday, September 6
MATH 54 with Ming Gu
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Student name:

You have 15 minutes to complete the quiz. Calculators are not permitted.

1. (4 points) If $A = \begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix}$, describe all solutions of $A\mathbf{x} = \mathbf{0}$ **in parametric vector form.**

$$\begin{aligned}
\begin{bmatrix} 3 & -6 & 6 \\ -2 & 4 & -2 \end{bmatrix} &\sim \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -2 \end{bmatrix} && (1)\\
&\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix} && (2)\\
&\sim \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix} && (3)\\
&\sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} && (4)
\end{aligned}$$

so we get the solution set $z = 0$, $y = t$ (free), and $x - 2y = 0$, so $x = 2t$. In parametric vector form, we can describe the set as

$$\left\{ \begin{bmatrix} 2t \\ t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

2. (4 points) Determine whether the vectors $\begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$ are linearly independent. Justify your answer.

Method 1: saying that the vectors are linearly independent is equivalent to saying that the system $\begin{bmatrix} -8 & 2 \\ 12 & -3 \\ -4 & -1 \end{bmatrix} \mathbf{x} = \mathbf{0}$ has only the trivial solution. But

$$\begin{aligned}
\begin{bmatrix} -8 & 2 \\ 12 & -3 \\ -4 & -1 \end{bmatrix} &\sim \begin{bmatrix} -4 & 1 \\ 4 & -1 \\ -4 & -1 \end{bmatrix} \\
&\sim \begin{bmatrix} -4 & 1 \\ 0 & 0 \\ 0 & -2 \end{bmatrix} \\
&\sim \begin{bmatrix} -4 & 1 \\ 0 & -2 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

the final form of which (not bothering to get it all the way to echelon form) has two pivot rows. Therefore the trivial solution is the only solution and the columns of our matrix (thus the vectors in question) are linearly independent.

Method 2: Same thing, just a little more direct. Suppose that there exist c_1 and c_2 such that

$$c_1 \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The first line of the equation implies that $c_1 = 4c_2$ but the third line implies that $c_1 = -4c_2$. Subtracting the second from the first gives $8c_2 = 0$, so $c_2 = 0$ and it follows that $c_1 = 0$. This means that the only linear combination that gives $\mathbf{0}$ is the trivial one, and so the vectors are linearly independent.

Alternate finish to method 2: Since both vectors are nonzero, there must be some c such that

$$c \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 12 \\ -4 \end{bmatrix}.$$

The first line impleis that $c = -4$ but the third implies that $c = 4$, which is a contradiction. Therefore no such c exists, the vectors are not parallel, and by a theorem in the book they are linearly independent.

3. (4 points) Mark each statement as True or False. You do not have to explain your reasoning.

   (a) If A is a 3×2 matrix, then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ cannot be one-to-one. FALSE: but it is true that the transformation cannot be onto.

   (b) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ always maps the origin of $\mathbb{R}^n$ to the origin of $\mathbb{R}^m$. TRUE: if you like, this can be proved using the basic properties of a linear transformation, since $A\mathbf{0} = A(\mathbf{0} + \mathbf{0}) = A\mathbf{0} + A\mathbf{0} = 2A\mathbf{0}$, which implies that $A\mathbf{0} = 0$.

   (c) If a set in $\mathbb{R}^n$ is linearly dependent, then the set contains more than n vectors. FALSE: it is true that any such set *must* be linearly dependent, but smaller sets can be linearly dependent as well.

   (d) If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent but $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly dependent then $\mathbf{z} \in \mathrm{Span}(\mathbf{x}, \mathbf{y})$. TRUE.