Quiz 12; Tuesday, November 29
MATH 54 with Ming Gu
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Student name:

You have 15 minutes to complete the quiz. Calculators are not permitted.

For any two piecewise continuous functions f and g on the interval $[-\pi, \pi]$, define $\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$ and $\|f\|^2 = \sqrt{\langle f, f\rangle}$.

1. (6 points) Find $\|\sin x\|$.

   ANSWER:

$$\begin{aligned}\|\sin x\|^2 &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x\,dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1-\cos 2x}{2}\,dx \\ &= \frac{1}{\pi}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) \|_{-\pi}^{\pi} \\ &= 1.\end{aligned}$$

   Therefore, $\|\sin x\| = \sqrt{1} = 1$.

2. (6 points) Show that $\sin 2x$ and $\cos 4x$ are orthogonal. (Hint: use what you know about odd and even functions.)

   ANSWER: $\sin 2x$ is odd and $\cos 4x$ is even, so $\sin(2x)\cos(4x)$ is odd. The integral of an odd function on the interval $[-\pi, \pi]$ is zero since the integral is symmetric about zero, and so the inner product is zero.

3. (Zero bonus points) Show that $\sin mx$ and $\sin nx$ are orthogonal whenever $m \neq n$.

   Use the two identities

$$\begin{aligned}\cos((m+n)x) &= \cos mx \cos nx - \sin mx \sin nx \\ \cos((m-n)x) &= \cos mx \cos nx + \sin mx \sin nx\end{aligned}$$

   Subtracting the second from the first and dividing by two gives the identity

$$\frac{\cos(m-n)x - \cos(m+n)x}{2} = \sin mx \sin nx.$$

   Therefore,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx &= \int_{-\pi}^{\pi} \frac{\cos(m-n)x - \cos(m+n)x}{2}\,dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} \cos(m-n)x\,dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(m+n)x\,dx.\end{aligned}$$

The second integral is always zero and the first one is zero provided $m \neq n$. Therefore, $\sin mx$ and $\sin nx$ are orthogonal (with respect to this inner product).