Quiz 10; Tuesday, November 15
MATH 54 with Ming Gu
GSI: Eric Hallman

Student name:

You have 15 minutes to complete the quiz. Calculators are not permitted.

1. (4 points) Verify that the vector functions $\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$ are solutions to the homogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \mathbf{x}$ on $\mathbb{R}$.

   ANSWER:

$$\mathbf{A}\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \begin{bmatrix} e^t \\ e^t \end{bmatrix} = \mathbf{x}_1'.$$

$$\mathbf{A}\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} \\ -3e^{-t} \end{bmatrix} = \mathbf{x}_2'.$$

2. (3 points) Verify that $\mathbf{x}_p = \begin{bmatrix} t \\ 2t-1 \end{bmatrix}$ is a particular solution to the nonhomogeneous system $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$, where $\mathbf{f}(t) = \begin{bmatrix} 0 \\ t \end{bmatrix}$.

   ANSWER:

$$\mathbf{A}\mathbf{x}_p + \mathbf{f}(t) = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} t \\ 2t-1 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{x}_p'.$$

3. (1 points) Write the general solution for the problem $\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$.

   ANSWER: $\mathbf{x} = \mathbf{x}_p + c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ for any $c_1, c_2 \in \mathbb{R}$.

4. (4 points) Demonstrate rigorously that $\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ 3e^{-t} \end{bmatrix}$ are linearly independent on $\mathbb{R}$.

   ANSWER: It suffices to check that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent in their first coordinate, so we just have to show that e^t and e^{-t} are linearly independent.

   Method 1: both are solutions to $(D+1)(D-1)y = 0$, so use the Wronskian.

   Method 2: show that they are linearly independent on the set $\{0, 1\}$ or something like that, which will show that they are LID on $\mathbb{R}$ as a whole.

   Method 3: if they were LD, then $e^t/e^{-t} = e^{2t}$ would be a constant function, but it isn't. By the contrapositive, they are linearly dependent.