Chapter 7.2-3: Probability, Bayes' Theorem

Monday, October 26

Warmup

1. Prove that $p(E \cup F) \le p(E) + p(F)$ for any events E and F.

$$p(E \cup F) = p(E) + p(F) - p(E \cap F) \ge p(E) + p(F) - 0 = p(E) + p(F).$$

2. Prove that $p(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^n p(E_i)$.

   Prove by induction: the case for $n = 2$ is already done above.

   Then if $p(\bigcup_{i=1}^n E_i) \le \sum_{i=1}^n p(E_i)$, we deduce that

$$\begin{aligned} p(\bigcup_{i=1}^{n+1} E_i) &= p(E_{n+1} \cup \bigcup_{i=1}^{n} E_i) \\ &\le p(E_{n+1}) + p(\bigcup_{i=1}^{n} E_i) \\ &\le p(E_{n+1}) = \sum_{i=1}^{n} p(E_i) \\ &= \sum_{i=1}^{n+1} p(E_i). \end{aligned}$$

   This completes the proof by induction.

3. Suppose that the probability of cold (below freezing) is 0.3 and the probability of precipitation (rain or snow) is 0.5. Assume the two events are independent.

   (a) What is the chance of snow?
   Since the events are independent, $p(C \cap P) = p(C)p(P) = (0.3)(0.5) = 0.15$.

   (b) What is the chance that it is cold but does not snow?
   $p(C \cap \overline{P}) = p(C)p(\overline{P}) = (0.3)(0.5) = 0.15$.

   (c) What is the chance of snow given that there is precipitation?
   $p(C \cap P|P) = p(C|P) = p(C) = 0.3$.

   (d) Even if the events are not independent, show that the chance of rain is at least 0.2.
   $p(\overline{C} \cap P) = p(\overline{C}) + p(P) - p(\overline{C} \cup P) \ge 0.7 + 0.5 - 1 = 0.2$.

Sampling With and Without Replacement

You have an urn containing 7 red balls and 5 green balls.

1. If you pick 5 balls at random, what is the chance of getting 3 red balls and 2 green balls?

$$\frac{\binom{7}{3}\binom{5}{2}}{\binom{12}{5}} \approx .442$$

2. What if you pick the balls one at a time, putting them back in the urn after each draw?

$$\binom{5}{3}(7/12)^3(5/12)^2 \approx .345$$

3. Suppose you want to draw 2 red balls and 1 green ball. Are you more likely to get this combination if you sample with or without replacement?

   The chance of drawing 2 red and 1 green without replacement is $\frac{\binom{7}{2}\cdot 5}{\binom{7}{3}} = 105/220 \approx .477$ and the chance with replacement is $3 \cdot (7/12)^2 \cdot (5/12) \approx .425$. The chances are greater if you draw without replacement.

Bayes Theorem

1. One urn has 7 red balls and 5 green balls. Another urn has 4 red balls and 9 green balls. If you pick an urn at random and draw a red ball, what is the chance that you picked the first urn?

   The odds are $(1/2)(7/12) : (1/2)(4/13) = 7/12 : 4/13 = 91 : 48$, so the chance of having picked the first urn is $91/139$.

2. If you draw 2 balls and pick one red and one green, what is the chance that it was from the first urn?

   Assume the draw was from a single urn, without replacement.

   The relative odds are $(1/2)7 \cdot 5/\binom{12}{2} : (1/2)4 \cdot 9/\binom{13}{2} = 2730 : 2376 = 455 : 396$, so the chance of having picked the first urn is $455/851$.

3. Now there are three urns, and the third has only green balls. If you pick an urn at random and draw a green ball, what is the chance that it was from the first urn?

   The relative odds are $(5/12)(1/3) : (9/13)(1/3) : 1(1/3) = 5/12 : 9/13 : 1 = 65 : 108 : 156$, so the probability of having picked the first urn is $65/(65 + 108 + 56) = 65/229 \approx .284$

Challenge

Two urns contain red and black balls. Urn A has 2 red and 1 black, and Urn B has 101 red and 100 black. An urn is chosen at random and you win a prize if you correctly name the urn on the basis of the evidence of two balls drawn from it. After the first is drawn and its color reported, you can decide whether or not the ball should be replaced before the second drawing. How should you decide?