Chapter 7.1-2: Probability!

Wednesday, October 21

Summary

- Probability of an event $E \subset S$: $p(E) = \sum_{s \in E} p(s)$.
- Uniform distribution: if $|S| = n$ then $p(s) = 1/n$ for all $s \in S$.
- $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
- Conditional probability: $p(E|F) = \frac{p(E \cap F)}{p(F)}$.
- E, F independent if $p(E \cap F) = p(E) \cdot p(F)$.
- $E_1, \ldots, E_n$ mutually independent if $p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{im}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$ for any subset $\{i_1, \ldots i_m\} \subset \{1, \ldots, n\}$.

Warmup

Suppose we roll a pair of fair six-sided dice.

1. What is the chance of rolling a seven?
2. What is the chance of *not* rolling a seven?
3. What is the chance of rolling an odd number *or* a number divisible by 3?
4. Say we roll the dice one at a time. If the first die lands on a 4, what is the chance that the second die will also land on a 4?

Uniform Probability

1. What is the probability that a 5-card hand drawn from a well-shuffled deck contains five cards of the same suit?

   There are $\binom{52}{5}$ hands in all. There are 4 ways to pick a suit and $\binom{13}{5}$ ways to pick five cards from that suit, so the probability is $\dfrac{4 \cdot \binom{13}{5}}{\binom{52}{5}}$.

2. Four friends are playing a game of Monopoly and want to roll the dice to decide who goes first. Find a way to decide who goes first that (1) requires only one roll of a pair of dice, and (2) gives every player an equal probability of going first.

   Roll the dice and decide who goes first based on the total rolled:

   8 or 9: player 1 goes first.

   5 or 6: player 2 goes first.

7 or 10: player 3 goes first.

2, 3, 4, 11, or 12: player 4 goes first. This distribution will give each player a 1/4 chance of going first.

3. Which is more likely: rolling an 8 when a total of 2 dice are rolled or rolling a total of 8 when three dice are rolled?

   The chance of rolling an 8 with 2 dice is 5/36. In the case of 3 dice: if the first die rolls a 1, 2, 3, 4, 5, or 6, then the next two dice must sum to 7, 6, 5, 4, 3, or 2, respectively, and there are $6+5+4+3+2+1$ ways for these scenarios to occur. The total number of ways to roll an 8 with 3 dice is therefore 21, and the probability of rolling an 8 is 21/216, which is less than 5/36.

4. Which is more likely: flipping exactly 5 heads out of 10 or flipping exactly 10 heads out of 20?

   The probability of flipping 5 heads out of 10 is $\binom{10}{5}/2^{10} \approx 24.6\%$ and the probability of flipping 10 heads out of 20 is $\binom{20}{10}/2^{20} \approx 17.6\%$. The first scenario is more likely.

   In fact, the chance of flipping n heads in $2n$ flips is strictly decreasing. To prove this, we will show that the chance of flipping n heads in $2n$ flips is greater than the chance of flipping $n+1$ heads in $2(n+1)$ flips:

$$
\begin{aligned}
4\binom{2n}{n} &> (\binom{2n}{n-1} + \binom{2n}{n}) + (\binom{2n}{n} + \binom{2n}{n+1}) \\
&= \binom{2n+1}{n} + \binom{2n+1}{n+1} \\
&= \binom{2n+2}{n+1} \\
\frac{4\binom{2n}{n}}{2^{2n+2}} &> \frac{\binom{2n+2}{n+1}}{2^{2n+2}} \\
\frac{\binom{2n}{n}}{2^{2n}} &> \frac{\binom{2n+2}{n+1}}{2^{2n+2}}
\end{aligned}
$$

Conditional Probability and Independence

1. You roll 2 dice. What is the probability that the sum is 7 if *at least* one of the two numbers is a 5?

   There are 2 ways to roll a 7 if at least one number is a 5 and there are 11 ways to roll at least one 5, so the probability is $2/11$.

2. You roll 2 dice. What is the probability that the sum is 7 if *exactly* one of the two numbers is a 5?

   There are 2 ways to roll a 7 if at least one number is a 5 and there are 10 ways to roll at least one 5, so the probability is $2/10 = 1/5$.

3. You flip 4 coins. What is the probability that you have flipped four heads if at least three of the coins landed on heads?

   There are 4 ways to flip exactly 3 heads and 1 way to flip 4 heads, so 5 ways in all. The chance is therefore $1/5$.

4. You flip 4 coins. What is the probability that you have flipped four heads if *the first* three coins landed on heads?

   $1/2$, since the chance that you have flipped 4 heads is now the same as the chance that the last flip was heads.

5. If E and F are independent events, prove or disprove that $\overline{E}$ and F are independent events.

   Proof: $p(F) = p((E \cup \overline{E}) \cap F) = p(E \cap F) + p(\overline{E} \cap F) = p(E)p(F)$, so $p(\overline{E} \cap F) = p(F) - p(E)p(F) = (1 - p(E))p(F) = p(\overline{E})p(F)$.

   This intuitively makes sense: for example, if I say "today is Friday" and that has no bearing on whether a coin flips heads or tails, then it should not matter either if I say "today is NOT Friday."

6. A three-member jury has two members who make the correct decision with probability p and one member who flips a fair coin. A one-member jury will make the correct decision with probability p. Which jury is more likely to make the correct decision?

   The chance that the three-member jury comes to the right conclusion is equal to the probability that exactly two of them make the right call (for all three possible pairs of jurors) plus the chance that all three will be right. If exactly two are right then the third must be wrong, so the chance is $p(1/2)(1-p) + p(1/2)p + (1-p)(1/2)p + p(1/2)p = (p/2)(1 - p + 1 - p + p + p) = p$.

   It appears that the two juries have the exact same chance of coming to the correct decision.

7. Two coins are flipped. Let A be the event that the first coin lands on heads, let B be the event that the second coin lands on heads, and let C be the event that an even number of heads is flipped. Show that A, B, and C are pairwise independent but not mutually independent.

   We can check that these events are pairwise independent because $p(A \cap B) = p(A)p(B) = 1/4$, and ditto for the pairs (A,C) and (B,C). They are not mutually independent because $p(A \cap B \cap C) = 0$ but $p(A)p(B)p(C) = 1/8$.