Chapter 6.5: Multinomials, Stars-and-Bars

Monday, October 19

Error Spotting

Of the solutions given to the following problems, determine which ones are correct and explain the errors in the incorrect ones.

1. How many strings of 5 digits (0-9) are there with at least one 7?

   (a) The number of strings with at least one 7 is equal to the total number of strings minus those with no 7s, which is $10^5 - 9^5$.
   CORRECT!

   (b) Fix one of the 5 digits and make it a 7. There are then 10 choices for each of the remaining digits, so the correct answer is $5 \cdot 10^4$.
   WRONG! This counts options like 77777 multiple times.

   (c) To get the number we add the number of strings with exactly 1, 2, 3, 4, and 5 sevens: thus the answer is $1 \cdot 9^4 + 1 \cdot 1 \cdot 9^3 + 1^3 \cdot 9^2 + 1^4 \cdot 9 + 1^5$.
   WRONG! This doesn't account for the choices we have in determining *which* spots should be 7s. A correct answer would be $5 \cdot 9^4 + \binom{5}{2} \cdot 9^3 + \binom{5}{3} \cdot 9^2 + 5 \cdot 9 + 1$.

2. How many strings of 5 digits have exactly two odd digits and exactly one digit greater than 5?

   (a) There are $\binom{5}{2}$ places to put the odd digits and 5 choices for each odd digit. Then there are 3 places to put the large digit and 4 choices (6,7,8,9) for that digit. The other two digits must be 0,2, or 4, so this gives $\binom{5}{2} \cdot 5^2 \cdot 3 \cdot 5 \cdot 3^2$ choices in all.
   WRONG! This does not account for the fact that the large digit might be one of the two odd digits.

   (b) If the digit greater than 5 is even then we have 5 choices each for the odd digits (1,3,5,7,9), 2 choices for the even digit (6,8), and three choices for the other two digits (0,2,4). If the large digit is odd then we have 3 choices for the small odd digit (1,3,5), two for the large odd digit (7,9), and three each for the other three digits (0,2,4). The answer is the sum of these two numbers times the 5! ways to permute the five digits, so $(5^2 \cdot 2 \cdot 3^2 + 3 \cdot 2 \cdot 3^3) \cdot 5!$
   Closer, but still WRONG! If some digits are repeated then there are not 5! distinct ways to permute them. We have to use the approach of the first wrong solution after splitting into the cases of whether the large digit is odd or even.

Miscellany

1. How many ways to put 7 identical balls in 4 identical bins?

   There are 11 ways in all:

$$\begin{aligned}
7+0+0+0&=7\\
6+1+0+0&=7\\
5+2+0+0&=7\\
5+1+1+0&=7\\
4+3+0+0&=7\\
4+2+1+0&=7\\
4+1+1+1&=7\\
3+3+1+0&=7\\
3+2+2+0&=7\\
3+2+1+1&=7\\
2+2+2+1&=7
\end{aligned}$$

2. How many ways to put 7 identical balls in 7 identical bins?

   15 ways total: all of the above ways, plus the following ones that would require more than 4 bins:

$$\begin{aligned}
3+1+1+1+1&=7\\
2+2+1+1+1&=7\\
2+1+1+1+1+1&=7\\
1+1+1+1+1+1+1&=7
\end{aligned}$$

3. How many ways to put 7 identical balls into a red bin, a blue bin, and a green bin?

   This is a stars-and-bars problem with 2 dividers between bins, so the answer is $\binom{7+2}{2} = 36$.

4. You have a 52-card deck. How many ways can you deal 5 cards to each of 6 players?

   $\dfrac{52!}{(5!)^6 \cdot 22!}$

5. How many ways can you deal 13 cards to each of 4 players?

   $\dfrac{52!}{13!13!13!13!}$

6. How many ways can you deal 13 cards to each of 4 players so that each player gets one card of each of the 13 values (ace-2-3-...-king)?

   Approach this problem value by value: there are 24 ways to distribute the aces to the 4 players (4!), 24 ways to distribute the twos, and so on, so the number of ways to deal the cards in this manner is $(4!)^{13} = 24^{13}$.

7. How many ways can you give 10 cookies to 4 friends if each friend gets at least 1 cookie?

   Give out 4 "necessary" cookies, then count the number of ways to give 6 cookies to 4 friends if some can get no cookies. 6 choices from 4 options with repetition, so the number of ways is $\dbinom{6+4-1}{4-1} = \dbinom{9}{3} = 84$.

8. How many 5-digit sequences have the digits in non-decreasing order?

   This is a stars-and-bars problem with 5 slots and 9 dividers between digits, so the answer is $\binom{14}{9} = 2002$

9. For how many 5-digit sequences are the digits in *strictly* increasing order (so no repetitions)?

   Just $\binom{10}{5}$: we can pick out the 5 digits first, then there is only one way to sort them.

10. How many ways can you split nine (non-identical) people into three (identical) groups of three?

    There are $\binom{9}{3,3,3} = 1680$ ways to pick the people for distinct teams, then 6 ways to rearrange the teams. The answer is 1680/6, or 280.

Poker

Say you draw a 5-card hand for a round of poker. You have a *full house* if you have three cards of one value and two of another (e.g. 5-5-5-7-7). You have a *straight* if the five cards are all consecutive values (e.g. A-2-3-4-5 or 10-J-Q-K-A, aces can count both low and high). You have a *flush* if all five cards are of the same suit (e.g. $3\heartsuit - 5\heartsuit - K\heartsuit - 10\heartsuit - 7\heartsuit$). A hand that is both a straight and a flush is a *straight flush*, and it is much more valuable than either a straight or a flush.
Your task: find the number of ways to draw a full house, straight, and flush (excluding straight flushes). Rank the three hands from most valuable to least valuable.

Good luck!