Chapter 6.3-6.4: Combinations and Permutations

Wednesday, October 14

Summary

- Pascal's Identity: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$
- Binomial Theorem: $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$
- Multinomial Theorem: $(x_1 + x_2 + \cdots + x_m)^n = \sum_{n_1+n_2+\cdots n_m = n} \frac{n!}{n_1! n_2! \cdots n_m!} x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$

Warmup

1. Six people are in a club. How many ways to choose three executive members? How many ways are there to choose a president and two vice presidents? Why is the second number divisible by the first?
2. Ten people are in a room, and everyone shakes everyone else's hand exactly once. How many handshakes were there?
3. Seven horses run a race. Assuming there are no ties, how many ways are there to hand out gold, silver, and bronze medals?
4. How many ways to hand out three medals if all of the medals are identical?
5. How many ways to hand out 1 gold, 1 silver, 1 bronze, and 4 participation medals?

Binomial and Multinomial Coefficients

1. How many distinct ways are there to arrange the letters in COUSCOUS?
2. How many distinct ways are there to arrange the letters in MISSISSIPPI?
3. If I flip a coin 7 times, how many ways are there to get an even number of heads?
4. If I roll five dice, how many ways are there to get 3 of one number and 2 of another?
5. If I draw 5 cards from a deck, how many ways are there (ignoring the order in which the cards are drawn) to draw 3 cards of one value and 2 of another?

Combinatorial Proofs

1. Show that if n is a positive integer then $\binom{2n}{2} = 2\binom{n}{2} + n^2$, by combinatorial proof and by algebraic manipulation. (Hint: there are n boys and n girls. If you want to pick 2 people for a team, break down by the number of girls you pick.)

Pascal's Triangle

1. Prove the *hockeystick identity*
$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$
using Pascal's identity (plus induction! For the induction, fix n arbitrarily and then use induction on r).

2. Illustrate the above identity on Pascal's Triangle, and use it to prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.